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On the Diophantine Equation $p^x + (p+14)^y = z^2$ where $p,\ p+14$ are Primes

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Abstract. In this paper, the Diophantine equation $p^x + (p+14)^y = z^2$, where p $p+14$ are primes and x, y, z are non-negative integers, is investigated. Some conditions for non-existence of the solutions of this equation are presented.

Keywords: Diophantine equation; Legendre symbol; Mihailescu's theorem

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1. Introduction

In recent papers, Diophantine equations of type $p^x + (p+h)^y = z^2$ where $p, p+h$ are primes and h is an even positive integer, have been studied. For instance, Bacani and Rabago [1] proved in 2015 that all non-negative integer solutions of the Diophantine equation $p^x + (p+2)^y = z^2$ where p and $p+2$ are primes, are $(x,y,z) = \left(1,1,\sqrt{2p+2}\right)$ where $\sqrt{2p+2}$ is an integer. In 2018, Burshtein [2,3] studied the Diophantine equation $p^x + (p+4)^y = z^2$ where $p>3$ and $p+4$ are primes, and the Diophantine equation $p^x + (p+6)^y = z^2$ where p and $p+6$ are primes with $x+y=2,3,4$. In the same year, Neres [9] showed that the Diophantine equation $p^x + (p+8)^y = z^2$ where $p>3$ and $p+8$ are primes, has no positive integer solution.

In 2019, Kumar, Gupta and Kishan [6] found that the Diophantine equation $p^x + (p+12)^y = z^2$ where p and $p+12$ are primes such that p is of the form $p=6n+1$, has no non-negative integer solution. In 2021, Dokchan and Pakapongpun [4] showed that the Diophantine equation $p^x + (p+20)^y = z^2$ where p and $p+20$ are primes, has no positive integer solution. In 2022, Tadee [12] studied the Diophantine equation $p^x + (p+10)^y = z^2$ where p and $p+10$ are primes.

In this paper, we will study the Diophantine equation $p^x + (p+14)^y = z^2$ where $p,\ p+14$ are primes and x, y, z are non-negative integers.

2. Preliminaries

In this section, we present some helpful Theorems and Lemmas.

Theorem 2.1. (Mihailescu's theorem) [7] The Diophantine equation $a^x - b^y = 1$ has the unique integer solution $(a,b,x,y) = (3,2,2,3)$ where a, b, x and y are integers with $\min\{a,b,x,y\} > 1$.

Theorem 2.2. [11] $(1,0,2)$ is the unique solution (x, y, z) for the Diophantine equation $3^x + 17^y = z^2$ where x, y and z are non-negative integers.

Lemma 2.1. Let q be prime. Then the Diophantine equation $1 + q^y = z^2$ has only two non-negative integer solutions $(q, y, z) \in \{(2,3,3),(3,1,2)\}$.

Proof: Let y and z be non-negative integers such that $1 + q^y = z^2$. It is easy to check that $z > 1$ and $y \neq 0$. If $y = 1$, then $(z-1)(z+1) = q$. Since q is prime, we have $z - 1 = 1$ and $z + 1 = q$. It implies that $z = 2$ and $q = 3$. Therefore $(q, y, z) = (3,1,2)$. If $y > 1$, then $\min\{z,q,2,y\} > 1$. By Theorem 2.1, it follows that $(q, y, z) = (2,3,3)$.

Definition 2.1. Let p be an odd prime and a be an integer such that $\gcd(a, p) = 1$. If the congruence $z^2 \equiv a \pmod{p}$ has an integer solution, then a is said to be a quadratic residue of p. Otherwise, a is called a quadratic non-residue of p.

Definition 2.2. Let p be an odd prime and a be an integer such that $\gcd(a, p) = 1$. The Legendre symbol, $\left(\frac{a}{p}\right)$, is defined by

$$\left(\frac{a}{p}\right) = \begin{cases} 1 & \text{if } a \text{ is a quadratic residue of } p \\ -1 & \text{if } a \text{ is a quadratic non-residue of } p. \end{cases}$$

Theorem 2.3. [10] Let p be an odd prime and a, b be integers with $\gcd(a, p) = 1$ and $\gcd(b, p) = 1$.

$$\left(\frac{ab}{p}\right) = \left(\frac{a}{p}\right)\left(\frac{b}{p}\right).$$

Lemma 2.2. [5] Let p be an odd prime.

$$\left(\frac{2}{p}\right) = \begin{cases} 1 & \text{if } p \equiv \pm 1 \pmod{8} \\ -1 & \text{if } p \equiv \pm 3 \pmod{8}. \end{cases}$$

Lemma 2.3. [8] Let p be an odd prime with $p \neq 7$.

$$\left(\frac{7}{p}\right) = \begin{cases} 1 & \text{if } p \equiv \pm1, \pm3, \pm9 \pmod{28} \\ -1 & \text{if } p \equiv \pm5, \pm11, \pm13 \pmod{28}. \end{cases}$$

By Lemma 2.2, Lamma 2.3 and Theorem 2.3, we have the following theorem.

Theorem 2.4. Let p be an odd prime with $p \neq 7$.

$$\left(\frac{14}{p}\right) = \begin{cases} 1 & \text{if } p \equiv \pm1, \pm5, \pm9, \pm11, \pm13, \pm25 \pmod{56} \\ -1 & \text{if } p \equiv \pm3, \pm15, \pm17, \pm19, \pm23, \pm27 \pmod{56}. \end{cases}$$

3. Main results

In this section, we present our results.

Theorem 3.1. Let p and $p+14$ be primes and x, y, z be non-negative integers.

(i) If x is even, then the Diophantine equation $p^x + (p+14)^y = z^2$ has no solution.

(ii) If y is even, then the Diophantine equation $p^x + (p+14)^y = z^2$ has the only one solution $(p, x, y, z) = (3, 1, 0, 2)$.

Proof:

(i) Let x be even. Then there exists a non-negative integer k such that $x = 2k$. Then $(z - p^k)(z + p^k) = (p+14)^y$. Since $p+14$ is prime, it implies that $z - p^k = (p+14)^u$ and $z + p^k = (p+14)^{y-u}$ for some non-negative integer u. Thus, $y \geq 2u$ and $2p^k = (p+14)^u\left((p+14)^{y-2u} - 1\right)$. Since p and $p+14$ are primes, we obtain that $u = 0$, and so $2p^k = (p+14)^y - 1$. It follows that $2p^k = (p+13)\left((p+14)^{y-1} + (p+14)^{y-2} + \cdots + 1\right)$. By Lemma 2.1, we have $k \neq 0$. Then $p + 13 = 2p^n$ for some positive integer n. Thus, $p \mid (p+13)$, and so $p = 13$. This is a contradiction since $p + 14 = 27$ is not prime.

(ii) Let y be even. Then there exists a non-negative integer l such that $y = 2l$. Then $\left(z - (p+14)^l\right)\left(z + (p+14)^l\right) = p^x$. Since p is prime, we get $z - (p+14)^l = p^v$ and $z + (p+14)^l = p^{x-v}$ for some non-negative integer v. Thus, $x \geq 2v$ and $2(p+14)^l = p^v\left(p^{x-2v} - 1\right)$. Since p and $p+14$ are primes, we have $v = 0$ and $2(p+14)^l = p^x - 1$. It follows that

$$2(p+14)^l = (p-1)\left(p^{x-1} + p^{x-2} + \cdots + 1\right).$$

Since $p+14$ is prime, there exists a non-negative integer m such that $p-1=2(p+14)^m$. Thus, $m=0$, and so $p=3$. By Theorem 2.2, we have $(p,x,y,z)=(3,1,0,2)$.

Corollary 3.1. Two Diophantine equations $25^x+19^y=z^2$ and $5^x+361^y=z^2$ have no non-negative integer solution.

Proof: Assume that a,b,c are non-negative integers such that $25^a+19^b=c^2$. It follows that $(x,y,z)=(2a,b,c)$ is a solution of the Diophantine equation $5^x+19^y=z^2$. This is a contradiction by Theorem 3.1. Similarly, to prove that the Diophantine equation $5^x+361^y=z^2$ has no solution.

Corollary 3.2. Let p and $p+14$ be primes with $p\equiv 3\pmod 4$. Then the Diophantine equation $p^x+(p+14)^y=z^{4n}$ has no solution, where n is a positive integer.

Proof: Assume that a,b,c are non-negative integers such that $p^a+(p+14)^b=c^{4n}$. Then $(x,y,z)=(a,b,c^{2n})$ is a non-negative integer solution of the Diophantine equation $p^x+(p+14)^y=z^2$. By Theorem 3.1, a and b are odd. Since $p\equiv 3\pmod 4$, we have $p\equiv 3,7\pmod 8$.

Case 1. $p\equiv 3\pmod 8$. Then $p+14\equiv 1\pmod 8$. Since a is odd, we get $3^a\equiv 3\pmod 8$, and so $c^{4n}=p^a+(p+14)^b\equiv 3^a+1\equiv 4\pmod 8$, a contradiction since $c^{4n}\equiv 0,1\pmod 8$.

Case 2. $p\equiv 7\pmod 8$. Since a and b are odd, we have $p^a\equiv(-1)^a\equiv -1\pmod 8$ and $(p+14)^b\equiv(-3)^b\equiv -3\pmod 8$, respectively. Thus, $c^{4n}=p^a+(p+14)^b\equiv 4\pmod 8$. This is impossible since $c^{4n}\equiv 0,1\pmod 8$.

Theorem 3.2. Let p and $p+14$ be primes with $p\equiv 6\pmod 7$. Then the Diophantine equation $p^x+(p+14)^y=z^2$ has no non-negative integer solution.

Proof: Assume that x,y,z are non-negative integers such that $p^x+(p+14)^y=z^2$. Since $p\equiv 6\pmod 7$, we get $p\neq 3$. By Theorem 3.1, x and y are odd. Since $p\equiv -1\pmod 7$, we obtain that $z^2=p^x+(p+14)^y\equiv(-1)^x+(-1)^y\equiv -2\pmod 7$. This is impossible since $z^2\equiv 0,1,2,4\pmod 7$.

By Theorem 3.2, we have the following corollary.

Corollary 3.3. The Diophantine equation $83^x+97^y=z^2$ has no non-negative integer solution.

Theorem 3.3. Let $p \neq 3$ and $p+14$ be primes with $p \equiv \pm 3, \pm 15, \pm 17, \pm 19, \pm 23, \pm 27 \pmod{56}$. Then the Diophantine equation $p^x + (p+14)^y = z^2$ has no non-negative integer solution.

Proof: Assume that x, y, z are non-negative integers such that $p^x + (p+14)^y = z^2$. By Theorem 3.1 and $p \neq 3$, it follows that x and y are odd. Then $z^2 \equiv 14^y \pmod{p}$, and so $\left(\frac{14^y}{p}\right) = 1$. Since $p+14$ is prime, we get $p \neq 2$ and $p \neq 7$. By Theorem 2.3, we have

$$\left(\frac{14}{p}\right) = \left(\frac{14}{p}\right)^y = \left(\frac{14^y}{p}\right) = 1,$$

which is impossible by Theorem 2.4.

4. Conclusion

We proved that the Diophantine equation $p^x + (p+14)^y = z^2$, where p and $p+14$ are primes, has no non-negative integer solution in the following cases: 1) when x is even, 2) when $p \equiv 6 \pmod{7}$, and 3) when $p \neq 3$ and $p \equiv \pm 3, \pm 15, \pm 17, \pm 19, \pm 23, \pm 27 \pmod{56}$.

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