On the Diophantine Equation $15^x - 13^y = z^2$

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Abstract. In this article, we prove that the Diophantine equation $15^x - 13^y = z^2$ has non-negative integer solution. The result reveals that the solution $(x, y, z) = (0, 0, 0)$.

Keywords: Diophantine equation; factoring method; modular arithmetic method

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1. Introduction

A popular topic in Mathematics is the Diophantine equation. This topic concerns finding a solution to an equation over an integer number. In [6], Mihailescu proved Catalan’s conjecture. This theorem is very important because it has been applied to prove many Diophantine equations. In [1], the Diophantine equation $2^x + 5^y = z^2$ was presented by Acu. He applied congruent and modular arithmetic theories to prove that the two solutions $(x, y, z)$ include $(3, 0, 3)$ and $(2, 1, 3)$. In [8], Suvarnamani et al. proved that $4^x + 7^y = z^2$ and $4^x + 11^y = z^2$ have no integer solution. Next, Chotchaisthit [5] demonstrated that $4^x + p^y = z^2$ where $p$ is any positive prime number have no solution. In 2018, Rabago [7] proved that $4^x - p^y = 3z^2$ where $p$ is prime has the set of all solutions $(x, y, z)$ including $(0, 0, 0)$ and $(q - 1, 1, 2^{q - 1} - 1)$ where $p = 2^q - 1$ and $q$ are prime. In 2019, Nechemia [3, 4] showed no solution to the Diophantine equation $7^x + 10^y = z^2$ when $x, y, z$ are positive integers, and presented the equation $6^x - 11^y = z^2$ when $x, y, z$ are positive integers. He suggested that the equation has one solution when $x = 2$, and no solution for $2 < x \leq 16$. Next, the Diophantine equation $2^x - 3^y = z^2$ was presented [9]. The authors proved that there are three solutions to the equation. Then, a group of researchers suggested that $p^x - 2^y = z^2$ where $p = k^2 + 2$ is a prime number has two solutions including $(x, y, z) = (0, 0, 0)$ or $(1, 1, k)$ [2]. After that, the Diophantine equation $7^x - 5^y = z^2$ was proved that the solution $(x, y, z)$ is $(0, 0, 0)$ [10]. Recently, the Diophantine equation $7^x - 2^y = z^2$ has been proved to have only one trivial solution


From the previous works, there is no general method to prove all sets of the Diophantine equation in the form \(a^n - b^n = z^2\). We still need to prove individual equations.

In this work, we study the Diophantine equation \(15^4 - 13^3 = z^2\). We use Modular Arithmetic to show all solutions to the equation.

2. Preliminaries

In this section, we introduce basic knowledge applying in the proof.

**Lemma 2.1.** For all \(n \in \mathbb{N}^+\). Then \(2^{2^n-1} \equiv 2, 5, 6, 7, 8 \text{ or } 11 (\text{mod } 13)\).

**Proof:** Let \(P(n) : 2^{2^n-1} \equiv 2, 5, 6, 7, 8 \text{ or } 11 (\text{mod } 13)\).

For \(n = 1\), we get \(2^{2(1)-1} = 2 \equiv 2 (\text{mod } 13)\). So \(P(1)\) is true.

We assume that \(P(k)\) is true for \(k \in \mathbb{N}^+\) that is

\[2^{2k-1} \equiv 2, 5, 6, 7, 8 \text{ or } 11 (\text{mod } 13)\].

Now, to prove that \(P(k+1)\) is true, we consider \(2^{2(k+1)-1} = 2^{2k+1} = 4 \cdot 2^{2k-1}\). Then we have

\[2^{2(k+1)-1} \equiv 4(2), 4(5), 4(6), 4(7), 4(8) \text{ or } 4(11) (\text{mod } 13)\]

\[\equiv 8, 20, 28, 32 \text{ or } 44 (\text{mod } 13)\]

\[\equiv 8, 7, 11, 2, 6 \text{ or } 5 (\text{mod } 13)\].

Hence

\[2^{2(k+1)-1} \equiv 2, 5, 6, 7, 8 \text{ or } 11 (\text{mod } 13)\].

Thus, \(P(k+1)\) is true. Therefore, by the Principle of Mathematical Induction, \(P(n)\) is true for all \(n \in \mathbb{N}^+\).

**Lemma 2.2.** For all \(x \in \mathbb{N}\). Then \(x^2 \equiv 0, 1, 3, 4, 9, 10 \text{ or } 12 (\text{mod } 13)\).

**Proof:** Let \(x \in \mathbb{N}\). There are \(q, r \in \mathbb{N}\) such that \(x = 13q + r\) for \(0 \leq r < 13\). It follows that \(x \equiv r (\text{mod } 13)\) and \(x^2 \equiv r^2 (\text{mod } 13)\).

Case 1: \(r = 0\), then we get \(x^2 \equiv 0 (\text{mod } 13)\).

Case 2: \(r = 1\) or 12, then we get \(x^2 \equiv 1 (\text{mod } 13)\).

Case 3: \(r = 2\) or 11, then we get \(x^2 \equiv 4 (\text{mod } 13)\).

Case 4: \(r = 3\) or 10, then we get \(x^2 \equiv 9 (\text{mod } 13)\).

Case 5: \(r = 4\) or 9, then we get \(x^2 \equiv 3 (\text{mod } 13)\).

Case 6: \(r = 5\) or 8, then we get \(x^2 \equiv 12 (\text{mod } 13)\).

Case 7: \(r = 6\) or 7, then we get \(x^2 \equiv 10 (\text{mod } 13)\).
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From all cases, we have $x^2 \equiv 0, 1, 3, 4, 9, 10 \text{ or } 12 \pmod{13}$.

\[ \square \]

### 3. Main result

**Theorem 3.1.** For all $x, y, z \in \mathbb{N}^+ \cup \{0\}$, the Diophantine equation $15^x - 13^y = z^2$ has a unique solution $(x, y, z) = (0, 0, 0)$.

**Proof:** Let $x, y, z \in \mathbb{N}^+ \cup \{0\}$ such that $15^x - 13^y = z^2$. (1)

The equation can be solved by considering the following four cases.

1. $x = 0$ and $y = 0$
2. $x = 0$ and $y > 0$
3. $x > 0$ and $y = 0$
4. $x > 0$ and $y > 0$.

**Case 1:** $x = 0$ and $y = 0$. It is easy to see that $z = 0$. We get the solution $(x, y, z) = (0, 0, 0)$.

**Case 2:** $x = 0$ and $y > 0$. The equation (1) becomes $1 - 13^y = z^2$. Because $1 - 13^y < 0$, we obtain $z^2 < 0$, impossible.

**Case 3:** $x > 0$ and $y = 0$. So (1) becomes $z^2 = 15^x - 1$. Because $15 \equiv 0 \pmod{3}$, this implies that $z^2 \equiv -1 \pmod{3}$ or $z^2 \equiv 2 \pmod{3}$, impossible.

**Case 4:** $x > 0$ and $y > 0$, we consider the following two subcases.

**Subcase 4.1:** $x$ is odd. By Lemma 2.1, it is easy to see that $2^x \equiv 2, 5, 6, 7, 8, 11 \pmod{13}$. By (1), we have $z^2 \equiv 2^x \pmod{13}$. This yields $z^2 \equiv 2, 5, 6, 7, 8, 11 \pmod{13}$.

**Subcase 4.2:** $x$ is even. Then $x = 2k$, $\exists k \in \mathbb{N}^+ \cup \{0\}$. It follows that $13^y = 15^k - z^2$. This is equivalent to $13^y = (15^k - z)(15^k + z)$. There are $\alpha$ and $\beta \in \mathbb{N}^+ \cup \{0\}$ such that $15^k - z = 13^\alpha$ and $15^k + z = 13^\beta$ where $\alpha < \beta$ and $\alpha + \beta = y$. This implies that $2 \cdot 15^k = 13^\alpha + 13^\beta$ or $2 \cdot 3^k \cdot 5^k = 13^\alpha \left(1 + 13^{\beta - \alpha}\right)$.

Since $13 \equiv 1 \pmod{3}$, (2) implies that $2 \equiv 0 \pmod{3}$. This is impossible. In all cases, it can be concluded that $(0, 0, 0)$ is a solution to the equation.

\[ \square \]

### 4. Conclusion

In this work, we have proved that the Diophantine equation $15^x - 13^y = z^2$ has a unique solution $(x, y, z) = (0, 0, 0)$. In the proof, we consider four cases, including cases 1: $x = 0$ and $y = 0$, case 2: $x = 0$ and $y > 0$, case 3: $x > 0$ and $y = 0$, and case 4: $x > 0$ and $y > 0$, and we use Modular Arithmetic. We obtain that the equation has only a trivial solution $(x, y, z) = (0, 0, 0)$. 

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REFERENCES