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Fixed Point Theorems for Sub–Compatible and Sub–Sequential Continuous Maps in *G*-Metric Space

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Abstract. The principal motive of this paper is to link several outcomes in the literature by discussing the phenomenon and discreteness of fixed points for fresh classes of mappings elucidated on a complete metric space. Specifically, we demonstrate common fixed point theorems for four self-maps in G–Metric Space employing the conceptions of Sub–Compatibility and Sub–Sequential Continuity.

Keywords: Sub-compatibility, sub-sequential continuity, fixed point, common fixed point theorem, *G*-Metric space

AMS Mathematics Subject Classification (2010): 47H10

1. Introduction

The Metric fixed point theory has a wide application in almost all fields of quantitative sciences, many authors have directed their attention to generalising the notion of a metric space. In this respect, several generalized metric spaces have come through by many authors in the last decade. Among all the generalized metric spaces, the notion of $G-$ metric space has attracted considerable attention from fixed point theorists. The concept of a $G-$ metric space was introduced by Mustafa and Sims [7], wherein the authors discussed the topological properties of this space and proved the analogue of the Banach contraction principle in the context of $G-$ metric spaces. Following these results, many authors like Ali et al. [1], Mustafa and Sims [9], Mustafa et al. [8], Ranjeth Kumar et al. [10], Saadati et al. [11], Shatanawi et al. [12,13,14], Mannro [4,5,6], Vishal and Raman [16] and Vishal and Tripathi [17] have studied and developed several common fixed point theorems in this framework. Considering the contemplations given by different researchers, the principal motive of this paper is to link several outcomes in the literature by discussing the phenomenon and discreteness of fixed points for fresh classes of mappings elucidated on a complete metric space. Specifically, we demonstrate common fixed point theorems for four self-maps in G–metric space employing the conceptions of sub–compatibility and sub–sequential continuity.

2. Preliminaries

Definition 2.1. Let X be a nonempty set, and let $G: X \times X \times X \to R^{+}$ be a function satisfying the following axioms:

(G_1) $G(x, y, z) = 0$ if $x = y = z$,

(G_2) $0 < G(x, x, y)$, for all $x, y \in X$ with $x \neq y$,

(G_3) $G(x, x, y) \leq G(x, y, z)$, for all $x, y, z \in X$ with $z \neq y$,

(G_4) $G(x, y, z) = G(x, z, y) = G(y, z, x) = \cdots$ (Symmetry in all three variables)

(G_5) $G(x, y, z) \leq G(x, a, a) + G(a, y, z), \forall\, x, y, z, a \in X$, (Rectangle inequality)
then the function G is called a generalised metric or more specifically, a $G-$metric on X and the pair (X, G) is called a $G-$metric space.

Definition 2.2. Let (X, G) be a $G-$metric space, and let $\{x_n\}$ be a sequence of points in X, a point $x \in X$ is said to be the limit of the sequence $\{x_n\}$ if $\lim_{m,n\to\infty} G(x, x_n, x_m) = 0$ and one says that sequence $\{x_n\}$ is $G-$convergent to x. So, that if $x_n \to x$ or $\lim_{n\to\infty} x_n \to x$ as $n \to \infty$ in a $G-$metric space (X, G) then for each $\in > 0$, there exists $k \in N$ such that $G(x, x_n, x_m) <\in$ for all $m, n \geq k$.

Proposition 2.1. Let (X, G) be a $G-$metric space. Then the following are equivalent:

(1) $\{x_n\}$ is $G-$convergent to x,
(2) $G(x_n, x_n, x) \to 0$ as $n \to \infty$,
(3) $G(x_n, x, x) \to 0$ as $n \to \infty$,
(4) $G(x_m, x_n, x) \to 0$ as $m, n \to \infty$.

Definition 2.3. Let (X, G) be a $G-$metric space. A sequence $\{x_n\}$ is called $G-$cauchy if , for each $\in > 0$ there exists $k \in N$ such that $G(x_n, x_m, x_l) <\in$ for all $n, m, l \geq k$ that is if $G(x_n, x_m, x_l) \to 0$ as $n, m, l \to \infty$.

Proposition 2.2. If (X, G) be a $G-$metric space. Then the following are equivalent:

(1) The sequence $\{x_n\}$ is $G-$cauchy,
(2) For each $\in > 0$, there exists $k \in N$ such that $G(x_n, x_m, x_m) <\in$ for all $n, m \geq k$.

Proposition 2.3. Let (X, G) be a $G-$metric space, then the function $G(x, y, z)$ is jointly continuous in all three of its variables.

Definition 2.4. A $G-$metric space (X, G) is called a symmetric $G-$metric space if $G(x, y, y) = G(y, x, x)$ for all $x, y \in X$.

Proposition 2.4. Every $G-$metric space (X, G) defines a metric space (X, d_G) by

(i) $d_G(x, y) = G(x, y, y) + G(y, x, x)$ for all $x, y \in X$.

If (X, G) is a symmetric G −metric space then

(ii) $d_G(x, y) = 2G(x, y, y)$ for all $x, y \in X$.

However, if (X, G) is not symmetric, then it follows from the G −metric properties that

(iii) $\frac{3}{2}G(x, y, y) \leq d_G(x, y) \leq 3G(x, y, y)$ for all $x, y \in X$.

Definition 2.5. A G −metric space (X, G) is said to be G −complete if every G −cauchy sequence in (X, G) is G −convergent in X.

Proposition 2.5. A G −metric space (X, G) is G −complete if and only if (X, d_G) is a complete metric space.

Proposition 2.6. Let (X, G) be a G −metric space, then for any $x, y, z, a \in X$ it follows that

(1) If $G(x, y, z) = 0$, then $x = y = z$,
(2) $G(x, y, z) \leq G(x, x, y) + G(x, x, z)$,
(3) $G(x, y, y) \leq 2G(y, x, x)$,
(4) $G(x, y, z) \leq G(x, a, z) + G(a, y, z)$,
(5) $G(x, y, z) \leq \frac{2}{3}\big(G(x, y, a) + G(x, a, z) + G(a, y, z)\big)$,
(6) $G(x, y, z) \leq \big(G(x, a, a) + G(y, a, a) + G(z, a, a)\big)$,

Definition 2.6. A pair of self mappings (A, B) of a G −metric space (X, G) is said to be compatible if $\lim_{n\to\infty} G(ABx_n, BAx_n, BAx_n) = 0$, whenever $\{x_n\}$ is a sequence in X such that $\lim_{n\to\infty} Ax_n = \lim_{n\to\infty} Bx_n = z$, where $z \in X$.

Definition 2.7. [3] Let A and B be self maps on X, then a point $x \in X$ is called a coincidence point of A and B if and only if $Ax = Bx$. In this case, $w = Ax = Bx$ is called a point of coincidence of A and B.

Definition 2.8. [3] Two self-mappings A and B on a metric space are said to be weakly compatible or coincidently commuting if they commute at their coincidence points, that are if $Au = Bu$ for some $u \in X$ then $ABu = BAu$.

Remark 2.1. It can be easily verified that compatible mappings are also weakly compatible but the converse is not necessarily true.

Definition 2.9. [3] Two self-mappings A and B of a metric space are said to be occasionally weakly compatible (owc) if and only if there exists a point $x \in X$ which is the coincidence point of A and B at which A and B commute.

Definition 2.10. [2] Let (X, G) be a G −metric space. Self maps A and B on X are said to be sub-compatible if and only if there exists a sequence $\{x_n\}$ in X such that $\lim_{n\to\infty} Ax_n =$

$$\lim_{n\to\infty} Bx_n = z, \quad \text{where } z \in X \text{ and satisfy}$$
$$\lim_{n\to\infty} G(ABx_n, BAx_n, BAx_n) = 0$$

Remark 2.2. From the above definitions, it is obvious that occasionally weakly compatible mappings are sub-compatible. However, in general, the converse is not true.

Definition 2.11. [15] Let (X, G) be a G −metric space. Self maps A and B on X are said to be reciprocal continuous if and only if

$$\lim_{n\to\infty} AB\, x_n = At \text{ and } \lim_{n\to\infty} BA\, x_n = Bt,$$

Whenever Sequence $\{x_n\}$ in X such that $\lim_{n\to\infty} A\, x_n = \lim_{n\to\infty} B\, x_n = t$, where $t \in X$. If A and B are both continuous then they are obviously reciprocally continuous but the converse is not necessarily true.

Definition 2.12. [15] Let (X, G) be a G −metric space. Self maps A and B on X are said to be sub-sequentially continuous if and only if there exists a sequence $\{x_n\}$ in X such that $\lim_{n\to\infty} A\, x_n = \lim_{n\to\infty} B\, x_n = t$, where $t \in X$ and satisfy $\lim_{n\to\infty} A\, Bx_n = At$ and $\lim_{n\to\infty} BA\, x_n = Bt$.

Remark 2.3. If two self-mappings A and B are continuous or reciprocally continuous, they are sub-sequentially continuous. However, in general, the converse is not true.

3. The main results

Theorem 2.1. Let A, B, S and T be four self-maps of a G − metric space (X, G). If the pairs (A, S) and (B, T) are sub-compatible and sub-sequentially continuous then

(1) A and S have a coincidence point;

(2) B and T have a coincidence point;

Further, let $\Phi: (R^+)^{18} \to R$ be an upper semi-continuous function satisfying the following condition:

(i) $\Phi(\mathrm{u, u, u, u, u, u}, 0,0,0,0,0,0, \mathrm{u, u, u, u, u, u}) > 0$, for all $u > 0$.

We suppose that (A, S) and (B, T) satisfy,

(ii)
$$\Phi\left\{\begin{matrix} G(Ax, By, By), G(Ax, Ty, Ty), \\ \frac{1}{2}\big(G(By, Ax, Ax) + G(By, Sx, Sx)\big), G(By, Ax, Ax), \\ \frac{1}{2}\big(G(Ax, By, By) + G(Ax, Ty, Ty)\big), G(By, Sx, Sx), \\ G(Ty, By, By), G(Ax, Sx, Sx), \\ \frac{1}{2}\big(G(Ty, By, By) + G(By, Ty, Ty)\big), G(By, Ty, Ty), \\ \frac{1}{2}\big(G(Ax, Sx, Sx) + G(Sx, Ax, Ax)\big), G(Sx, Ax, Ax), \\ G(Sx, Ty, Ty), G(Ty, Ax, Ax), \\ \frac{1}{2}\big(G(Sx, Ty, Ty) + G(Sx, By, By)\big), G(Ty, Sx, Sx), \\ \frac{1}{2}\big(G(Ty, Ax, Ax) + G(Ty, Sx, Sx)\big), G(Sx, By, By) \end{matrix}\right\} \le 0$$

for all $x, y \in X$.

Then A, B, S and T have a unique common fixed point.

Proof: Since the pairs (A, S) and (B, T) are sub-compatible and sub-sequentially continuous, there exist two sequences $\{x_n\}$ and $\{y_n\}$ in X such that

$\lim_{n\to\infty} Ax_n = \lim_{n\to\infty} Sx_n = z$, where $z \in X$ and which satisfy

$$\lim_{n\to\infty} G\,(ASx_n, SAx_n, SAx_n) = G\,(Az, Sz, Sz) = 0$$

and $\lim_{n\to\infty} By_n = \lim_{n\to\infty} Ty_n = z'$, where $z' \in X$ and which satisfy

$$\lim_{n\to\infty} G\,(BTy_n, TBy_n, TBy_n) = G\,(Bz', Tz', Tz') = 0$$

Therefore, $Az = Sz$ and $Bz' = Tz'$, that is z is a coincidence point of A and S and z' is a coincidence point of B and T.

Now, we prove that $z = z'$.

Indeed by inequality (ii), we have

$$\Phi\left\{\begin{matrix} G(Ax_n, By_n, By_n), G(Ax_n, Ty_n, Ty_n), \\ \frac{1}{2}\big(G(By_n, Ax_n, Ax_n) + G(By_n, Sx_n, Sx_n)\big), G(By_n, Ax_n, Ax_n), \\ \frac{1}{2}\big(G(Ax_n, By_n, By_n) + G(Ax_n, Ty_n, Ty_n)\big), G(By_n, Sx_n, Sx_n), \\ G(Ty_n, By_n, By_n), G(Ax_n, Sx_n, Sx_n), \\ \frac{1}{2}\big(G(Ty_n, By_n, By_n) + G(By_n, Ty_n, Ty_n)\big), G(By_n, Ty_n, Ty_n), \\ \frac{1}{2}\big(G(Ax_n, Sx_n, Sx_n) + G(Sx_n, Ax_n, Ax_n)\big), G(Sx_n, Ax_n, Ax_n), \\ G(Sx_n, Ty_n, Ty_n), G(Ty_n, Ax_n, Ax_n), \\ \frac{1}{2}\big(G(Sx_n, Ty_n, Ty_n) + G(Sx_n, By_n, By_n)\big), G(Ty_n, Sx_n, Sx_n), \\ \frac{1}{2}\big(G(Ty_n, Ax_n, Ax_n) + G(Ty_n, Sx_n, Sx_n)\big), G(Sx_n, By_n, By_n) \end{matrix}\right\} \le 0$$

Since Φ is upper semi−continuous, taking the limit as $n \to \infty$ yields

$$\Phi\left\{\begin{matrix} G(z, z', z'), G(z, z', z'), \\ \frac{1}{2}\big(G(z', z, z) + G(z', z, z)\big), G(z', z, z), \\ \frac{1}{2}\big(G(z, z', z') + G(z, z', z')\big), G(z', z, z), \\ G(z', z', z'), G(z, z, z), \\ \frac{1}{2}\big(G(z', z', z') + G(z', z', z')\big), G(z', z', z'), \\ \frac{1}{2}\big(G(z, z, z) + G(z, z, z)\big), G(z, z, z), \\ G(z, z', z'), G(z', z, z), \\ \frac{1}{2}\big(G(z, z', z') + G(z, z', z')\big), G(z', z, z), \\ \frac{1}{2}\big(G(z', z, z) + G(z', z, z)\big), G(z, z', z') \end{matrix}\right\} \le 0$$

which contradicts (i), if $z \neq z'$. Hence $z = z'$

Also, we claim that $Az = z$.

If $Az \neq z$, using (ii), we get

$$\Phi\left\{\begin{array}{c} G(Az, By_n, By_n), G(Az, Ty_n, Ty_n), \\ \frac{1}{2}\big(G(By_n, Az, Az) + G(By_n, Sz, Sz)\big), G(By_n, Az, Az), \\ \frac{1}{2}\big(G(Az, By_n, By_n) + G(Az, Ty_n, Ty_n)\big), G(By_n, Sz, Sz), \\ G(Ty_n, By_n, By_n), G(Az, Sz, Sz), \\ \frac{1}{2}\big(G(By_n, Ty_n, Ty_n) + G(Ty_n, By_n, By_n)\big), G(By_n, Ty_n, Ty_n), \\ \frac{1}{2}\big(G(Az, Sz, Sz) + G(Sz, Az, Az)\big), G(Sz, Az, Az), \\ G(Sz, Ty_n, Ty_n), G(Ty_n, Az, Az), \\ \frac{1}{2}\big(G(Sz, Ty_n, Ty_n) + G(Sz, By_n, By_n)\big), G(Ty_n, Sz, Sz), \\ \frac{1}{2}\big(G(Ty_n, Az, Az) + G(Ty_n, Sz, Sz)\big), G(Sz, By_n, By_n) \end{array}\right\} \leq 0$$

Since Φ is upper semi−continuous, taking the limit as $n \to \infty$ yields

$$\Phi\left\{\begin{array}{c} G(Az, z, z), G(Az, z, z), \\ \frac{1}{2}\big(G(z, Az, Az) + G(z, Az, Az)\big), G(z, Az, Az), \\ \frac{1}{2}\big(G(Az, z, z) + G(Az, z, z)\big), G(z, Az, Az), \\ G(z, z, z), G(Az, Az, Az), \\ \frac{1}{2}\big(G(z, z, z) + G(z, z, z)\big), G(z, z, z), \\ \frac{1}{2}\big(G(Az, Az, Az) + G(Az, Az, Az)\big), G(Az, Az, Az), \\ G(Az, z, z), G(z, Az, Az), \\ \frac{1}{2}\big(G(Az, z, z) + G(Az, z, z)\big), G(z, Az, Az), \\ \frac{1}{2}\big(G(z, Az, Az) + G(z, Az, Az)\big), G(Az, z, z) \end{array}\right\} \leq 0$$

$$\Phi\left\{\begin{array}{c} G(Az, z, z), G(Az, z, z), G(z, Az, Az), G(z, Az, Az), G(Az, z, z), \\ G(z, Az, Az), 0,0,0,0, \\ 0,0, G(Az, z, z), G(z, Az, Az), G(Az, z, z), \\ G(z, Az, Az), G(z, Az, Az), G(Az, z, z) \end{array}\right\} \leq 0$$

This contradicts (i). Hence $z = Az = Sz$.

Again, suppose that $Bz \neq z$, using (ii), we get

$$\Phi\left\{\begin{matrix} G(Az,Bz,Bz), G(Az,Tz,Tz), \\ \frac{1}{2}\big(G(Bz,Az,Az)+G(Bz,Sz,Sz)\big), G(Bz,Az,Az), \\ \frac{1}{2}\big(G(Az,Bz,Bz)+G(Az,Tz,Tz)\big), G(Bz,Sz,Sz), \\ G(Tz,Bz,Bz), G(Az,Sz,Sz), \\ \frac{1}{2}\big(G(Tz,Bz,Bz)+G(Bz,Tz,Tz)\big), G(Bz,Tz,Tz), \\ \frac{1}{2}\big(G(Az,Sz,Sz)+G(Sz,Az,Az)\big), G(Sz,Az,Az), \\ G(Sz,Tz,Tz), G(Tz,Az,Az), \\ \frac{1}{2}\big(G(Sz,Tz,Tz)+G(Sz,Bz,Bz)\big), G(Tz,Sz,Sz), \\ \frac{1}{2}\big(G(Tz,Az,Az)+G(Tz,Sz,Sz)\big), G(Sz,Bz,Bz) \end{matrix}\right\} \leq 0$$

$$\Phi\left\{\begin{matrix} G(z,Bz,Bz), G(z,Bz,Bz), \\ \frac{1}{2}\big(G(Bz,z,z)+G(Bz,z,z)\big), G(Bz,z,z), \\ \frac{1}{2}\big(G(z,Bz,Bz)+G(z,Bz,Bz)\big), G(Bz,z,z), \\ G(Bz,Bz,Bz), G(z,z,z), \\ \frac{1}{2}\big(G(Bz,Bz,Bz)+G(Bz,Bz,Bz)\big), G(Bz,Bz,Bz), \\ \frac{1}{2}\big(G(z,z,z)+G(z,z,z)\big), G(z,z,z), \\ G(z,Bz,Bz), G(Bz,z,z), \\ \frac{1}{2}\big(G(z,Bz,Bz)+G(z,Bz,Bz)\big), G(Bz,z,z), \\ \frac{1}{2}\big(G(Bz,z,z)+G(Bz,z,z)\big), G(z,Bz,Bz) \end{matrix}\right\} \leq 0$$

$$\Phi\left\{\begin{matrix} G(Az,z,z), G(Az,z,z), G(z,Az,Az), G(z,Az,Az), G(Az,z,z), \\ G(z,Az,Az), 0,0,0,0, \\ 0,0, G(Az,z,z), G(z,Az,Az), \mathrm{G}(Az,z,z), \\ G(z,Az,Az), G(z,Az,Az), G(Az,z,z) \end{matrix}\right\} \leq 0$$

This contradicts (i). Hence $z = Bz = Tz$.

Therefore $z = Az = Bz = Sz = Tz$; that is z is a common fixed point of A, B, S and T.
For Uniqueness. Suppose that there exist another fixed point w of A, B, S and T such that $z \neq w$. Then by condition (ii), we have

$$\Phi\left\{\begin{array}{c} G(Az,Bw,Bw),G(Az,Tw,Tw), \\ \frac{1}{2}\big(G(Bw,Az,Az)+G(Bw,Sz,Sz)\big),G(Bw,Az,Az), \\ \frac{1}{2}\big(G(Az,Bw,Bw)+G(Az,Tw,Tw)\big),G(Bw,Sz,Sz), \\ G(Tw,Bw,Bw),G(Az,Sz,Sz), \\ \frac{1}{2}\big(G(Tw,Bw,Bw)+G(Bw,Tw,Tw)\big),G(Bw,Tw,Tw), \\ \frac{1}{2}\big(G(Az,Sz,Sz)+G(Sz,Az,Az)\big),G(Sz,Az,Az), \\ G(Sz,Tw,Tw),G(Tw,Az,Az), \\ \frac{1}{2}\big(G(Sz,Tw,Tw)+G(Sz,Bw,Bw)\big),G(Tw,Sz,Sz), \\ \frac{1}{2}\big(G(Tw,Az,Az)+G(Tw,Sz,Sz)\big),G(Sz,Bw,Bw) \end{array}\right\}\leq 0$$

$$\Phi\left\{\begin{array}{c} G(z,w,w),G(z,w,w), \\ \frac{1}{2}\big(G(w,z,z)+G(w,z,z)\big),G(w,z,z), \\ \frac{1}{2}\big(G(z,w,w)+G(z,w,w)\big),G(w,z,z), \\ G(w,w,w),G(z,z,z), \\ \frac{1}{2}\big(G(w,w,w)+G(w,w,w)\big),G(w,w,w), \\ \frac{1}{2}\big(G(z,z,z)+G(z,z,z)\big),G(z,z,z), \\ G(z,w,w),G(w,z,z), \\ \frac{1}{2}\big(G(z,w,w)+G(z,w,w)\big),G(w,z,z), \\ \frac{1}{2}\big(G(w,z,z)+G(w,z,z)\big),G(z,w,w) \end{array}\right\}\leq 0$$

$$\Phi\left\{\begin{array}{c} G(z,w,w),G(z,w,w),G(w,z,z),G(w,z,z),G(z,w,w), \\ G(w,z,z),0,0,0,0, \\ 0,0,G(z,w,w),G(w,z,z),G(z,w,w), \\ G(w,z,z),G(w,z,z)G(z,w,w) \end{array}\right\}\leq 0$$

This contradicts condition (i). Hence $z=w$. Therefore z is a unique common fixed point of A,B,S and T.

Theorem 2.2. Let A,B,S and T be four self maps of a $G-$metric space (X,G). If the pairs (A,S) and (B,T) are sub-compatible and sub-sequentially continuous then

(1) A and S have a coincidence point;
(2) B and T have a coincidence point;

Further, let $\Phi: (R^{+})^{15} \to R$ be an upper semi-continuous function satisfying the following condition:

(i) $\Phi(\mathrm{u,u,u,u,u,0,0,0,0,0,u,u,u,u,u}) > 0$, for all $u > 0$.

We suppose that (A,S) and (B,T) satisfy,

$$\text{(ii)}\quad \Phi\left\{\begin{array}{c} G(Ax,By,By), G(Ax,Ty,Ty), G(By,Ax,Ax), \\ \frac{1}{2}\big(G(Ax,By,By)+G(Ax,Ty,Ty)\big), G(By,Sx,Sx), \\ G(Ty,By,By), G(Ax,Sx,\ Sx), G(By,Ty,Ty), \\ \frac{1}{2}\big(G(Ax,Sx,\ Sx)+G(Sx,Ax,Ax)\big), G(Sx,Ax,Ax), \\ G(Sx,Ty,Ty), G(Ty,Ax,Ax), G(Ty,Sx,Sx), \\ \frac{1}{2}\big(G(Ty,Ax,Ax)+G(Ty,Sx,Sx)\big), G(Sx,By,By) \end{array}\right\} \leq 0$$

for all $x, y \in X$.

Then A, B, S and T have a unique common fixed point.

Proof: Since the pairs (A,S) and (B,T) are sub-compatible and sub-sequentially continuous, there exist two sequences $\{x_n\}$ and $\{y_n\}$ in X such that

$\lim_{n\to\infty} Ax_n = \lim_{n\to\infty} Sx_n = z,$ where $z \in X$ and which satisfy

$$\lim_{n\to\infty} G\,(ASx_n, SAx_n, SAx_n) = G\,(Az, Sz, Sz) = 0$$

and $\lim_{n\to\infty} By_n = \lim_{n\to\infty} Ty_n = z'$, where $z' \in X$ and which satisfy

$$\lim_{n\to\infty} G\,(BTy_n, TBy_n, TBy_n) = G\,(Bz', Tz', Tz') = 0$$

Therefore, $Az = Sz$ and $Bz' = Tz'$, that is z is a coincidence point of A and S and z' is a coincidence point of B and T.

Now, we prove that $z = z'$.

Indeed by inequality (ii), we have

$$\Phi\left\{\begin{array}{c} \mathrm{G(Ax_n,By_n,By_n), G(Ax_n,Ty_n,Ty_n), G(By_n,Ax_n,Ax_n),} \\ \frac{1}{2}\big(\mathrm{G(Ax_n,By_n,By_n)+G(Ax_n,Ty_n,Ty_n)}\big), \mathrm{G(By_n,Sx_n,Sx_n),} \\ \mathrm{G(Ty_n,By_n,By_n), G(Ax_n,Sx_n,Sx_n), G(By_n,Ty_n,Ty_n),} \\ \frac{1}{2}\big(\mathrm{G(Ax_n,Sx_n,Sx_n)+G(Sx_n,Ax_n,Ax_n)}\big), \mathrm{G(Sx_n,Ax_n,Ax_n),} \\ \mathrm{G(Sx_n,Ty_n,Ty_n), G(Ty_n,Ax_n,Ax_n), G(Ty_n,Sx_n,Sx_n),} \\ \frac{1}{2}\big(\mathrm{G(Ty_n,Ax_n,Ax_n)+G(Ty_n,Sx_n,Sx_n)}\big), \mathrm{G(Sx_n,By_n,By_n)} \end{array}\right\} \leq 0$$

Since Φ is upper semi–continuous, taking the limit as $n \to \infty$ yields

$$\Phi\left\{\begin{array}{c} G(z,z',z'),G(z,z',z'),G(z',z,z), \\ \frac{1}{2}\big(G(z,z',z')+G(z,z',z')\big),G(z',z,z), \\ G(z',z',z'),G(z,z,z),G(z',z',z'), \\ \frac{1}{2}\big(G(z,z,z)+G(z,z,z)\big),G(z,z,z), \\ G(z,z',z'),G(z',z,z),G(z',z,z), \\ \frac{1}{2}\big(G(z',z,z)+G(z',z,z)\big),G(z,z',z') \end{array}\right\}\leq 0$$

which contradicts (i), if $z\neq z'$. Hence $z=z'$
Also, we claim that $Az=z$. If $Az\neq z$, using (ii), we get

$$\Phi\left\{\begin{array}{c} G(Az,By_n,By_n),G(Az,Ty_n,Ty_n),G(By_n,Az,Az), \\ \frac{1}{2}\big(G(Az,By_n,By_n)+G(Az,Ty_n,Ty_n)\big),G(By_n,Sz,Sz), \\ G(Ty_n,By_n,By_n),G(Az,Sz,Sz),G(By_n,Ty_n,Ty_n), \\ \frac{1}{2}\big(G(Az,Sz,Sz)+G(Sz,Az,Az)\big),G(Sz,Az,Az), \\ G(Sz,Ty_n,Ty_n),G(Ty_n,Az,Az),G(Ty_n,Sz,Sz), \\ \frac{1}{2}\big(G(Ty_n,Az,Az)+G(Ty_n,Sz,Sz)\big),G(Sz,By_n,By_n) \end{array}\right\}\leq 0$$

Since Φ is upper semi−continuous, taking the limit as $n\to\infty$ yields

$$\Phi\left\{\begin{array}{c} G(Az,z,z),G(Az,z,z),G(z,Az,Az), \\ \frac{1}{2}\big(G(Az,z,z)+G(Az,z,z)\big),G(z,Az,Az), \\ G(z,z,z),G(Az,Az,Az),G(z,z,z), \\ \frac{1}{2}\big(G(Az,Az,Az)+G(Az,Az,Az)\big),G(Az,Az,Az), \\ G(Az,z,z),G(z,Az,Az),G(z,Az,Az), \\ \frac{1}{2}\big(G(z,Az,Az)+G(z,Az,Az)\big),G(Az,z,z) \end{array}\right\}\leq 0$$

$$\Phi\left\{\begin{array}{c} G(Az,z,z),G(Az,z,z),G(z,Az,Az),G(Az,z,z),G(z,Az,Az), \\ 0,0,0,0,0,G(Az,z,z),G(z,Az,Az),G(z,Az,Az),G(z,Az,Az), \\ G(Az,z,z) \end{array}\right\}\leq 0$$

This contradicts (i). Hence $z=Az=Sz$.

Again, suppose that $Bz\neq z$, using (ii), we get

$$\Phi\left\{\begin{matrix} G(Az,Bz,Bz),G(Az,Tz,Tz),G(Bz,Az,Az), \\ \frac{1}{2}\big(G(Az,Bz,Bz)+G(Az,Tz,Tz)\big),G(Bz,Sz,Sz), \\ G(Tz,Bz,Bz),G(Az,Sz,Sz),G(Bz,Tz,Tz), \\ \frac{1}{2}\big(G(Az,Sz,Sz)+G(Sz,Az,Az)\big),G(Sz,Az,Az), \\ G(Sz,Tz,Tz),G(Tz,Az,Az),G(Tz,Sz,Sz), \\ \frac{1}{2}\big(G(Tz,Az,Az)+G(Tz,Sz,Sz)\big),G(Sz,Bz,Bz) \end{matrix}\right\}\leq 0$$

$$\Phi\left\{\begin{matrix} G(z,Bz,Bz),G(z,Bz,Bz),G(Bz,z,z), \\ \frac{1}{2}\big(G(z,Bz,Bz)+G(z,Bz,Bz)\big),G(Bz,z,z), \\ G(Bz,Bz,Bz),G(z,z,z),G(Bz,Bz,Bz), \\ \frac{1}{2}\big(G(z,z,z)+G(z,z,z)\big),G(z,z,z), \\ G(z,Bz,Bz),G(Bz,z,z),G(Bz,z,z), \\ \frac{1}{2}\big(G(Bz,z,z)+G(Bz,z,z)\big),G(z,Bz,Bz) \end{matrix}\right\}\leq 0$$

$$\Phi\left\{\begin{matrix} G(z,Bz,Bz),G(z,Bz,Bz),G(Bz,z,z),G(z,Bz,Bz),G(Bz,z,z), \\ 0,0,0,0,0,G(z,Bz,Bz),G(Bz,z,z),G(Bz,z,z),G(Bz,z,z) \\ G(z,Bz,Bz) \end{matrix}\right\}\leq 0$$

This contradicts (i).
Hence $z = Bz = Tz$.
Therefore $z = Az = Bz = Sz = Tz$; that is z is a common fixed point of A, B, S and T.
For Uniqueness. Suppose that there exist another fixed point w of A, B, S and T such that $z \neq w$. Then by condition (ii), we have

$$\Phi\left\{\begin{matrix} G(Az,Bw,Bw),G(Az,Tw,Tw),G(Bw,Az,Az), \\ \frac{1}{2}\big(G(Az,Bw,Bw)+G(Az,Tw,Tw)\big),G(Bw,Sz,Sz), \\ G(Tw,Bw,Bw),G(Az,Sz,Sz),G(Bw,Tw,Tw), \\ \frac{1}{2}\big(G(Az,Sz,Sz)+G(Sz,Az,Az)\big),G(Sz,Az,Az), \\ G(Sz,Tw,Tw),G(Tw,Az,Az),G(Tw,Sz,Sz), \\ \frac{1}{2}\big(G(Tw,Az,Az)+G(Tw,Sz,Sz)\big),G(Sz,Bw,Bw) \end{matrix}\right\}\leq 0$$

$$\Phi\left\{\begin{array}{c} G(z,w,w),G(z,w,w),G(w,z,z), \\ \frac{1}{2}\big(G(z,w,w)+G(z,w,w)\big),G(w,z,z), \\ G(w,w,w),G(z,z,z),G(w,w,w), \\ \frac{1}{2}\big(G(z,z,z),+G(z,z,z)\big),G(z,z,z), \\ G(z,w,w),G(w,z,z),G(w,z,z), \\ \frac{1}{2}\big(G(w,z,z)+G(w,z,z)\big),G(z,w,w) \end{array}\right\}\leq 0$$

$$\Phi\left\{\begin{array}{c} G(z,Bz,Bz),G(z,Bz,Bz),G(Bz,z,z),G(z,Bz,Bz),G(Bz,z,z), \\ 0,0,0,0,0,G(z,Bz,Bz),G(Bz,z,z),G(Bz,z,z),G(Bz,z,z) \\ G(z,Bz,Bz) \end{array}\right\}\leq 0$$

This contradicts condition (i).
Hence $z = w$.
Therefore z is a unique common fixed point of A, B, S and T.

4. Conclusions

Considering the contemplations given by different researchers, the principal motive of this chapter is to link several outcomes in the literature by discussing the phenomenon and discreteness of fixed points for fresh classes of mappings elucidated on a complete metric space. Specifically, we demonstrate common fixed point theorems for four self-maps in G–Metric Space, employing sub–compatibility and sub–sequential Continuity. This can be further extended for more number of self−mappings satisfying a more complex class of inequality.

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