Annals of Pure and Applied Mathematics
Vol. 27, No. 2, 2023, 51-54
ISSN: 2279-087X (P), 2279-0888(online)
Published on 20 May 2023
www.researchmathsci.org
DOI: http://dx.doi.org/10.22457/apam.v27n2a02898

Annals of
Pure and Applied
Mathematics

The Largest Eigenvalue of the Signless Laplacian and Some Hamiltonian Properties of Graphs

Rao Li

Department of Computer Science, Engineering, and Mathematics
University of South Carolina Aiken, Aiken, SC 29801, USA.
E-mail: raol@usca.edu

Received 12 December 2022; accepted 15 April 2023

Abstract. In this note, we present sufficient conditions based on the largest eigenvalue of the signless Laplacian for Hamiltonian and traceable graphs.

Keywords: The largest eigenvalue of the signless Laplacian, Hamiltonian graph, traceable graph

AMS Mathematics Subject Classification (2010): 05C45, 05C50

1. Introduction

We consider only finite undirected graphs without loops or multiple edges. Notation and terminology not defined here follow those in [1]. For a graph G, we use n to denote its order $|V(G)|$. The minimum degree and connectivity of a graph G are denoted by $\delta(G)$ and $\kappa(G)$, respectively. A subset V_1 of the vertex set V(G) of G is independent if no two vertices in V_1 are adjacent in G. We define E(X, Y) as $\{e : e = xy \in E, x \in X, y \in Y \}$. A graph G is semiregular if G is bipartite and all the vertices in the same part of bipartition have the same degree. The signless Laplacian of a graph G, denoted Q(G), is defined as D(G) + A(G), where D(G) is a diagonal matrix whose entries are the degrees of vertices in G and A(G) is the adjacency matrix of G. We use $q_1(G)$ to denote the largest eigenvalue of Q(G). A cycle C in a graph G is called a Hamiltonian cycle of G if C contains all the vertices of G. A graph G is called Hamiltonian if G has a Hamiltonian cycle. A path P in a graph G is called a Hamiltonian path of G if P contains all the vertices of G. A graph G is called traceable if G has a Hamiltonian path. In this note, we present sufficient conditions involving the largest eigenvalue of the signless Laplacian, minimum degree, and connectivity for the Hamiltonian and traceable graphs. The main results are as follows.

Theorem 1.1. Let G be a graph of order $n \geq 3$ and e edges with connectivity κ ($\kappa \geq 2$). If

$$q_1 \leq (\kappa + 1)\, \delta^2/e + e/(n - \kappa - 1),$$

then G is Hamiltonian or G is $K_{\kappa, \kappa + 1}$.

Theorem 1.2. Let G be a graph of order $n \geq 12$ and e edges with connectivity κ ($\kappa \geq 1$). If $q_1 \leq (\kappa + 2)\, \delta^2/e + e/(n - \kappa - 2)$, then G is traceable or G is $K_{\kappa, \kappa + 2}$.

2. Lemmas

We need the following results as our lemmas when we prove Theorem 1.1 and Theorem 1.2.

Lemma 2.1 below is from [4].

Lemma 2.1. Let G be a balanced bipartite graph of order 2n with bipartition (A, B). If d(x) + d(y) $\geq$ n + 1 for any x $\in$ A and any y $\in$ B with xy $\notin$ E, then G is Hamiltonian.

Lemma 2.2 below is from [2].

Lemma 2.2. Let G be a 2-connected bipartite graph with bipartition (A, B), where |A| $\geq$ |B|. If each vertex in A has degree at least k and each vertex in B has degree at least i, then G contains a cycle of length at least 2min(|B|, k + i - 1, 2k - 2).

Lemma 2.3 below is from [5].

Lemma 2.3. Let G be a graph with at least one edge, then

$$q_1 \geq \sum_{v \in V} d^2(v)/e.$$

3. Proofs

Proof of Theorem 1.1. Let G be a graph satisfying the conditions in Theorem 1.1. Suppose, to the contrary, that G is not Hamiltonian. Then n $\geq$ 2κ + 1 (otherwise $\delta \geq \kappa \geq n/2$ and G is Hamiltonian). Since $\kappa \geq 2$, G has a cycle. Choose a longest cycle C in G and give an orientation on C. Since G is not Hamiltonian, there exists a vertex $u_0 \in V(G) - V(C)$. By Menger's theorem, we can find s ($s \geq \kappa$) pairwise disjoint (except for u_0) paths $P_1, P_2, ..., P_s$ between u_0 and V(C). Let v_i be the end vertex of P_i on C, where $1 \leq i \leq s$. Without loss of generality, we assume that the appearance of $v_1, v_2, ..., v_s$ agrees with the orientation of C. We use v_i^+ to denote the successor of v_i along the orientation of C, where $1 \leq i \leq s$. Since C is a longest cycle in G, we havt $v_i^+ \neq v_{i+1}$, where $1 \leq i \leq s$ and the index s + 1 is regarded as 1. Moreover, $\{u_0, v_1^+, v_2^+, ..., v_s^+\}$ is independent (otherwise, G would have cycles which are longer than C). Set S := $\{u_0, v_1^+, v_2^+, ..., v_\kappa^+\}$. Then S is independent. Let $u_i = v_i^+$ for each i with $1 \leq i \leq \kappa$. Set T := V - S = $\{w_1, w_2, ..., w_{n-\kappa-1}\}$. Some ideas in [3] will be used below. Notice that $\sum_{u \in S} d(u)$ = |E(S, V - S)| $\leq \sum_{w \in V-S} d(w)$ and $\sum_{u \in S} d(u) + \sum_{w \in V-S} d(w)$ = 2e, we have that

$$\sum_{u \in S} d(u) \leq e \leq \sum_{w \in V-S} d(w).$$

By the conditions of Theorem 1.1, Lemma 2.3, and Cauchy-Schwarz inequality, we have

$$\begin{aligned}(\kappa + 1)\, \delta^2/e + e/(n - \kappa - 1) &\geq q_1 \\ &\geq \sum_{v \in V} d^2(v)/e \\ &\geq \sum_{u \in S} d^2(u)/e + \sum_{w \in V-S} d^2(w)/e \\ &\geq (\kappa + 1)\, \delta^2/e + (\sum_{w \in V-S} d(w))^2/(e(n - \kappa - 1)) \\ &\geq (\kappa + 1)\, \delta^2/e + e/(n - \kappa - 1).\end{aligned}$$

Thus all the inequalities above become equalities. Therefore $d(u_0) = d(u_1) = \ldots = d(u_{\kappa+1})$

$= \delta$, $d(w_0) = d(w_1) = \ldots = d(w_{n-\kappa-1}) := \delta_1$, and $e = \sum_{u \in S} d(u) = \sum_{w \in V-S} d(w)$. Since $e = \sum_{u \in S} d(u)$, there is no edge between any pair of vertices in V - S. So V - S is independent. Notice again that

$$e = \sum_{u \in S} d(u) = (\kappa + 1)\,\delta = \sum_{w \in V-S} d(w) = (n - \kappa - 1)\,\delta_1 \geq (n - \kappa - 1)\,\delta,$$

we have that $n \leq 2\kappa + 2$. Since $n - \kappa - 1 = |V - S| \geq d(u_0) \geq \delta \geq \kappa$, we have that $n \geq 2\kappa + 1$. Hence $n = 2\kappa + 1$ or $n = 2\kappa + 2$.

When $n = 2\kappa + 1$, then $n - \kappa - 1 = \kappa$. Since $d(u_i) = \delta \geq \kappa$ for i with $0 \leq i \leq \kappa$, $u_iw_j \in E$ for each i with $0 \leq i \leq \kappa$ and for each j with $1 \leq j \leq n - \kappa - 1$. Hence G is $K_{\kappa, \kappa+1}$.

When $n = 2\kappa + 2$, then $n - \kappa - 1 = \kappa + 1$ and G is a balanced bipartite graph. From Lemma 2.1, we have G is Hamiltonian, a contradiction.

This completes the proof of Theorem 1.1. ■

Proof of Theorem 1.2. Let G be a graph satisfying the conditions in Theorem 1.2. Suppose, to the contrary, that G is not traceable. Then $n \geq 2\kappa + 2$ (otherwise $\delta \geq \kappa \geq (n - 1)/2$ and G is traceable). Choose a longest path P in G and give an orientation on P. Let x and y be the two end vertices of P. Since G is not traceable, there exists a vertex $u_0 \in V(G) - V(P)$. By Menger's theorem, we can find s ($s \geq \kappa$) pairwise disjoint (except for u_0) paths $P_1, P_2, \ldots, P_s$ between u_0 and V(P). Let v_i be the end vertex of P_i on P, where $1 \leq i \leq s$. Without loss of generality, we assume that the appearance of $v_1, v_2, \ldots, v_s$ agrees with the orientation of P. Since P is the longest path in G, $x \neq v_i$ and $y \neq v_i$. for each i with $1 \leq i \leq s$, otherwise G would have paths which are longer than P. We use v_i^+ to denote the successor of v_i along the orientation of P, where $1 \leq i \leq s$. Since P is the longest path in G, we have that $v_i^+ \neq v_{i+1}$, where $1 \leq i \leq s - 1$. Moreover, $\{u_0, v_1^+, v_2^+, \ldots, v_s^+, x\}$ is independent (otherwise, G would have paths which are longer than P). Set $S := \{u_0, v_1^+, v_2^+, \ldots, v_\kappa^+, x\}$. Then S is independent. Let $u_i = v_i^+$ for each i with $1 \leq i \leq \kappa$ and $u_{\kappa+1} = x$. Set $T := V - S = \{w_1, w_2, \ldots, w_{n-\kappa-2}\}$. Some ideas in [3] will be used below. Notice that

$$\sum_{u \in S} d(u) = |E(S, V - S)| \leq \sum_{w \in V-S} d(w)$$

and

$$\sum_{u \in S} d(u) + \sum_{w \in V-S} d(w) = 2e,$$

we have that

$$\sum_{u \in S} d(u) \leq e \leq \sum_{w \in V-S} d(w).$$

By the conditions of Theorem 1.2, Lemma 2.3, and Cauchy-Schwarz inequality, we have

$$\begin{aligned}(\kappa + 2)\,\delta^2/e + e/(n - \kappa - 2) &\geq q_1 \\ &\geq \sum_{v \in V} d^2(v)/e \\ &\geq \sum_{u \in S} d^2(u)/e + \sum_{w \in V-S} d^2(w)/e \\ &\geq (\kappa + 2)\,\delta^2/e + \left(\sum_{w \in V-S} d(w)\right)^2/(e(n - \kappa - 2)) \\ &\geq (\kappa + 2)\,\delta^2/e + e/(n - \kappa - 2).\end{aligned}$$

Thus all the inequalities above become equalities. Therefore $d(u_0) = d(u_1) = \ldots = d(u_{\kappa + 1}) = \delta$, $d(w_0) = d(w_1) = \ldots = d(w_{n - \kappa - 2}) := \delta_1$, and $e = \sum_{u \in S} d(u) = \sum_{w \in V-S} d(w)$. Since $e = \sum_{u \in S} d(u)$, there is no edge between any pair of vertices in V - S. So V - S is independent. Notice again that $e = \sum_{u \in S} d(u) = (\kappa + 2)\,\delta = \sum_{w \in V-S} d(w) = (n - \kappa - 2)\,\delta_1 \geq (n - \kappa - 2)\,\delta$, we have that $n \leq 2\kappa + 4$. Since $n - \kappa - 2 = |V - S| \geq d(u_0) \geq \delta \geq \kappa$, we have that $n \geq 2\kappa + 2$.

Hence $n = 2\kappa + 2$, $n = 2\kappa + 3$, or $n = 2\kappa + 4$.

When $n = 2\kappa + 2$, then $n - \kappa - 2 = \kappa$. Since $d(u_i) = \delta \geq \kappa$ for i with $0 \leq i \leq \kappa + 1$, $u_iw_j \in E$ for each i with $0 \leq i \leq \kappa + 1$ and for each j with $1 \leq j \leq n - \kappa - 2$. Hence G is $K_{\kappa, \kappa + 2}$.

When $n = 2\kappa + 3$, then $n - \kappa - 2 = \kappa + 1$. Since $n = 2\kappa + 3 \geq 12$, $\kappa \geq 5$. Notice that each vertex in S or T has a degree at least $\delta \geq \kappa$. From Lemma 2.2, we have G has a cycle of length $2\kappa + 2$. Since $n = 2\kappa + 3$ and $\kappa \geq 5$, G has a path containing all the vertices of G. Namely, G is traceable, a contradiction.

When $n = 2\kappa + 4$, then $n - \kappa - 2 = \kappa + 2$. Since $n = 2\kappa + 4 \geq 12$, $\kappa \geq 4$. Notice that each vertex in S or T has degree at least $\delta \geq \kappa$. From Lemma 2.2, we have G has a cycle of length $2\kappa + 4$, which implies that G is traceable, a contradiction.

This completes the proof of Theorem 1.2. ■

4. Conclusion

In this note, we present new sufficient conditions involving the largest eigenvalue of the signless Laplacian, minimum degree, and connectivity for Hamiltonian and traceable graphs.

Acknowledgements. The author would like to thank the referee for his/her suggestions and comments, which helped to improve the original manuscript.

Conflict of interest. As a sole author, there is no question of conflict of interest.

Authors' Contributions. The author did all the work for this paper.

REFERENCES

1. J.A.Bondy and U.S.R.Murty, *Graph Theory with Applications,* Macmillan, London and Elsevier, New York (1976).
2. B.Jackson, Long cycles in bipartite graphs, *Journal of Combinatorial Theory Series B,* 38 (1985) 118-131.
3. R.Li, The largest eigenvalue conditions for Hamiltonian and traceable graphs, *Discrete Mathematics Letters,* 12 (2023) 50-53.
4. J.Moon and L.Moser, On Hamiltonian bipartite graphs, *Israel J. Math.,* 1 (1963) 163-165.
5. B.Zhou, Signless Laplacian spectral radius and Hamiltonicity, *Linear Algebra and its Applications,* 432 (2010) 566-570.