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Radio Labeling of Double Triangular Snake Graphs

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Abstract. One important problem in graph theory is graph coloring or graph labeling. Labeling problem is a well-studied problem due to its wide applications, especially in frequency assignment in(mobile) communication systems, coding theory, x-ray crystallography, radar, circuit design, etc. For two non-negative integers, labeling of a graph is a function from the node set to the set of non-negative integers. For any two distinct vertices x and y of G, a radio labeling is an injective function $h: V(G) \rightarrow N \cup \{0\}$ such that $d(x,y) + |h(x) - h(y)| \geq 1 + D$ where D is the diameter of G. The radio number of $h, rn(h)$ is the maximum value of $rn(h)$ taken overall radio labelings h of G. This paper determines the radio number of double triangular snake graphs.

Keywords: Radio labeling, Radio number, Triangular snake, Double triangular snake.

AMS Mathematics Subject Classification (2010): 90B05

1. Introduction

The graphs considered here are finite, undirected and simple. Let $V(G)$ and $E(G)$ denote the vertex and edge sets of G. A labeling of a graph G is an assignment of integers to the vertices, edges, or both subject to certain conditions. In 1980, Hale [5] introduced the notion of Radio labeling. In 1988, Roberts suggested a solution for the channel assignment problem. Chartrand et al. [4] investigated the radio number for paths and cycles and were completely solved by Liu and Zhu [3]. The span of a labeling h is the maximum integer that h maps to a vertex of G. The radio number of $G, rn(G)$, is the lowest span taken overall radio labeling of the graph G. The distance between two vertices x and y of G is denoted by $d(x,y)$. We follow J.A.Bondy [2] and Gallian[1] for standard terminology and notations. Some basic results and definitions are taken from [6] - [12].

Definition 1.1. *A snake graph G is a connected planar graph consisting of a finite sequence of tiles $G_1, G_2, \ldots, G_d$, such that G_i and G_{i+1} share exactly one edge e_i and this edge is either the north edge of G_i and the south edge of G_{i+1} or the east edge of G_i and the west edge of G_{i+1}. Denote by Int$(G) = \{e_1, e_2, \ldots, e_{d-1}\}$ the set of interior edges of the snake graph G.*

Example 1.2.

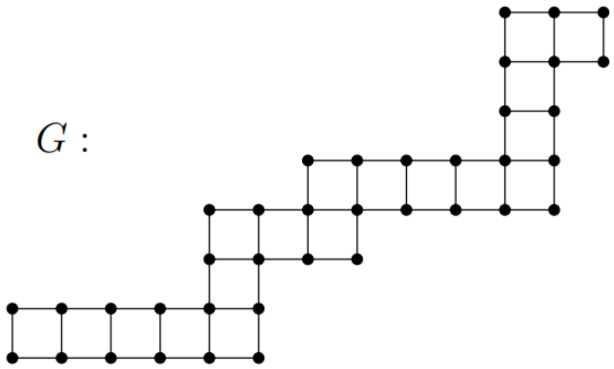

Figure 0: Sanke graph

Definition 1.3. *[3] The distance* $d(x, y)$ *from a vertex* x *to a vertex* y *in a connected graph* G *is the minimum length of the* $x - y$ *paths in* G.

Definition 1.4. *[4] The eccentricity* $e(x)$ *of a vertex* x *in a connected graph* G *is the distance between* x *and a vertex farthest from* x *in* G.

Definition 1.5. *[10] The diameter* D *is the greatest eccentricity among the vertices of* G.

Definition 1.6. *[6] A triangular snake* T_n *is obtained from a path* $x_1, x_2, x_3, \dots, x_n$ *by joining* x_i *and* x_{i+1} *to a new vertex* y_i *for* $1 \leq i \leq n - 1$.

Definition 1.7. *A Double Triangular Snake consists of two triangular snakes that have a common path. That is, a double triangular snake* DT_n *is a graph obtained from a path* $x_1, x_2, \dots, x_n$ *by joining* x_i *and* x_{i+1} *to two new vertices* y_i *and* $z_i, 1 \leq i \leq n - 1$.

Definition 1.8. *[4] The level function* $L: V(G) \to \{0,1,2, \dots, N\}$ *with respect to a centre vertex* z *by* $L(x) = min\{d(z, x)/z \in V(C(G)), x \in V(G)\}$ *where* $V(C(G))$ *is the vertices of the centre of a graph* G.

Definition 1.9. *[4] The total level of a graph* G, *denoted by* $L(G)$, *is defined as*

$$L(G) = \sum_{x \in V(G)} L(x)$$

2. Observations

For the double triangular snake graph DT_n with diameter D, we have the following observations.

The total number of vertices of the double triangular snake graph DT_n is

$$|V(DT_n)| = p = 3n - 2 \tag{2.1}$$

The distance between any two vertices of the double triangular snake graph DT_n is

$$d(x,y) \le \begin{cases} L(x)+L(y) & \text{if } n \text{ is odd} \\ L(x)+L(y)+1, & \text{if } n \text{ is even} \end{cases} \tag{2.2}$$

The total level of the double triangular snake graph DT_n is

$$L(DT_n) = \begin{cases} \frac{3}{4}(n^2+1) & \text{if } n \text{ is odd} \\ \frac{3}{4}n(n-2), & \text{if } n \text{ is even} \end{cases} \tag{2.3}$$

The level function with respect to the vertices x_1 and x_p is

$L(x_1)=0; L(x_p)=\frac{n-1}{2}$ when n is odd and $L(x_1)=0; L(x_p)=0$ when n is even (2.4)

3. Main results

In this section, we proved one Lemma and determine two results of the radio number of double triangular snake graphs.

Lemma 3.1. *Let $\{x_1,x_2,\dots,x_p\}$ be the ordering of $V(DT_n)$ such that $h(x_i)<h(x_{i+1})$. Define $h: V(DT_n)\to N$ by $h(x_1)=0$, $h(x_{i+1})=h(x_i)+D+1-d(x_i,x_{i+1})$ and $d(x_i,x_{i+1})\le a+1$ where $a=\lfloor\frac{n}{2}\rfloor, 1\le i\le p-1$. Then h is a radio labeling.*

Proof: Let h be an assignment of distinct non-negative integers to $V(DT_n)$ such that $h(x_1)=0$ and $h(x_{i+1})=h(x_i)+D+1-d(x_i,x_{i+1})$ and $d(x_i,x_{i+1})\le a+1, 1\le i\le p-1$ where $a=\lfloor\frac{n}{2}\rfloor$. Let $h_i=h(x_{i+1})-h(x_i), 1\le i\le p-1$.

Now, we want to prove that h is a radio labeling. That is, for any $i\ne j, d(x_i,x_j)+|h(x_j)-h(x_i)|\ge 1+D$. Without loss of generality, let $j\ge i+2$ then

$h(x_j)-h(x_i)=h_i+h_{i+1}+\dots+h_{j-1}$

$=(D+1)-d(x_i,x_{i+1})+(d+1)-d(x_{i+1},x_{i+2})+\dots+(D+1)-d(x_{j-1},x_j)$

$=(j-i)(D+1)-d(x_i,x_{i+1})-d(x_{i+1},x_{i+2})-\dots-d(x_{j-1},x_j)$

$=(j-i)(D+1)-(j-i)(a+1)$

as $d(x_i,x_{i+1})\le a+1$

Let $n=2a+1$. In this case, diameter $D=2a$

$h(x_j)-h(x_i)\ge(j-i)(2a+1)-(j-i)(a+1)$

$=(j-i)(2a+1-a-1)=(j-i)(a)$

$\ge 2a$ as $j\ge i+2$

$=D+1-d(x_i,x_j)$ as $d(x_i,x_j)\ge 1$

Let $n=2a$. In this case $D=2a-1$.

If $j-i=$ even then

$$h(x_j)-h(x_i)\ge(j-i)(D+1)-\left(\frac{j-i}{2}\right)(a+1)-\left(\frac{j-i}{2}\right)(a)$$

$$=(j-i)(2a)-\left(\frac{j-i}{2}\right)(2a+1)=(j-i)\left[a-\frac{1}{2}\right]$$

$$\ge 2\left(a-\frac{1}{2}\right)=2a-1=D+1-1$$

$$\ge D+1-d(x_i,x_j) \text{ as } d(x_i,x_j)\ge 1$$

If $j-i=$ odd then

$$h(x_j) - h(x_i) \geq (j-i)(D+1) - \left(\frac{j-i-1}{2}\right)(a+1) - \left(\frac{j-i-1}{2}\right)(a)$$
$$\geq D + 1 - d(x_i, x_j)$$

Hence $d(x_i, x_j) + |h(x_j) - h(x_i)| \geq 1 + D$.

Thus, h is a radio labeling.

Theorem 3.2. *Let DT_n be a Double Triangular snake graph on n vertices. Then $rn(DT_n) = \frac{3n^2-5n+2}{2}$ if n is odd.*

Proof: Let h be an optimal radio labeling for DT_n and $\{x_1, x_2, \ldots, x_p\}$ be the ordering of $V(DT_n)$ suchthat $0 = h(x_1) < h(x_2) < \ldots < h(x_p)$. Then $h(x_{i+1}) - h(x_i) \geq (D+1) - d(x_i, x_{i+1})$ for all $1 \leq i \leq p-1$.

Summing up these $p-1$ inequalities, we get

$$\sum_{i=1}^{p-1} [h(x_{i+1}) - h(x_i)] \geq \sum_{i=1}^{p-1} (D+1) - \sum_{i=1}^{p-1} d(x_i, x_{i+1})$$
$$h(x_p) - h(x_1) \geq (p-1)(D+1) - \sum_{i=1}^{p-1} d(x_i, x_{i+1})$$
$$h(x_p) \geq (p-1)(D+1) - \sum_{i=1}^{p-1} d(x_i, x_{i+1})$$

Therefore, $rn(DT_n) = h(x_p) \geq (p-1)(D+1) - \sum_{i=1}^{p-1} d(x_i, x_{i+1})$ (3.1)

Let $n = 2a + 1$ and $a = \lfloor\frac{n}{2}\rfloor, a \geq 1$

In this case, diameter $D = 2a$ and $p = 3n - 2$

From (2.2), we have

$$d(x_i, x_{i+1}) \leq L(x_i) + L(x_{i+1}), 1 \leq i \leq p-1$$
$$\sum_{i=1}^{p-1} d(x_i, x_{i+1}) \leq \sum_{i=1}^{p-1} [L(x_i) + L(x_{i+1})]$$
$$= [L(x_1) + L(x_2) + \ldots + L(x_{p-1})] + [L(x_2) + L(x_3) + \ldots + L(x_p)]$$
$$= \sum_{x \in V(DT_n)} L(x) - L(x_p) + \sum_{x \in V(DT_n)} L(x) - L(x_1)$$
$$= 2\sum_{x \in V(DT_n)} L(x) - L(x_1) - L(x_p) = 2L(DT_n) - L(x_1) - L(x_p)$$

$$= 2 \times \frac{3}{4}(n^2 - 1) - \left(\frac{n-1}{2}\right)$$ [choosing $x_1 \in V(C(DT_n))$,

$$L(x_1) = 0, L(x_p) = \frac{n-1}{2}] = \frac{3}{2}(n^2 - 1) - \left(\frac{n-1}{2}\right) = \frac{3n^2-n-2}{2} \qquad (3.2)$$

substituting (3.2) in (3.1), we get

$$rn(DT_n) = h(x_p) \geq (p-1)(D+1) - \left(\frac{3n^2-n-2}{2}\right)$$
$$= (3n-3)(n-1+1) - \left(\frac{3n^2-n-2}{2}\right)$$
$$= (3n-3)n - \left(\frac{3n^2-n-2}{2}\right)$$
$$= \frac{3n^2-5n+2}{2}$$

Define a function $h: V(DT_n) \to \{0,1,2, \ldots, \frac{3n^2-5n+2}{2}\}$ by $h(x_1) = 0$ and

$$h(x_{i+1}) = h(x_i) + D + 1 - d(x_i, x_{i+1}), \text{ for } 1 \leq i \leq p-1$$

Now, we label the vertices in the ordering as follows.

Let V'_c be the centre of DT_n. Let $V_{Li}^j, i = 1,2,\ldots,a, j = 1,2,3$ be the vertices on the left side while $V_{Ri}^j, i = 1,2,\ldots,a, j = 1,2,3$ be the vertices on the right side with respect to the centre V'_c of DT_n. Let $\{x_1, x_2, \ldots, x_p\}$ be the ordering of the vertices of DT_n. Label the vertices $x_1, x_2, \ldots, x_p$ as in the following procedure.

$$V'_c \to V_{Ra}^2 \to V_{L1}^3 \to V_{Ra}^1 \to V_{L1}^2 \to V_{Ra}^3 \to V_{L1}^1$$
$$V_{R(a-1)}^2 \to V_{L2}^3 \to V_{R(a-1)}^1 \to V_{L2}^2 \to V_{R(a-1)}^3 \to V_{L2}^1$$
$$V_{R(a-2)}^2 \to V_{L3}^3 \to V_{R(a-2)}^1 \to V_{L3}^2 \to V_{R(a-2)}^3 \to V_{L3}^1$$
....
$$V_{R1}^2 \to V_{La}^3 \to V_{R1}^1 \to V_{La}^2 \to V_{R1}^3 \to V_{La}^1$$

Thus, it is possible to assign labeling to the vertices of DT_n with span equal to the lower bound and satisfy the condition of lemma 3.1; hence h is a radio labeling.

Thus, we have $rn(DT_n) \leq \frac{3n^2-5n+2}{2}$

Hence, $rn(DT_n) = \frac{3n^2-5n+2}{2}$

Example 3.3. *In Table 1, Figure 1, Figure 2 and Figure 3 an ordering of the vertices, renamed version, ordering version and optimal radio labeling for DT_9 are shown.*

$$V_c^1 \to V_{R4}^2 \to V_{L1}^3 \to V_{R4}^1 \to V_{L1}^2 \to V_{R4}^3 \to V_{L1}^1$$
$$V_{R3}^2 \to V_{L2}^3 \to V_{R3}^1 \to V_{L2}^2 \to V_{R3}^3 \to V_{L2}^1$$
$$V_{R2}^2 \to V_{L3}^3 \to V_{R2}^1 \to V_{L3}^2 \to V_{R2}^3 \to V_{L3}^1$$
$$V_{R1}^2 \to V_{L4}^3 \to V_{R1}^1 \to V_{L4}^2 \to V_{R1}^3 \to V_{L4}^1$$

Table 1:

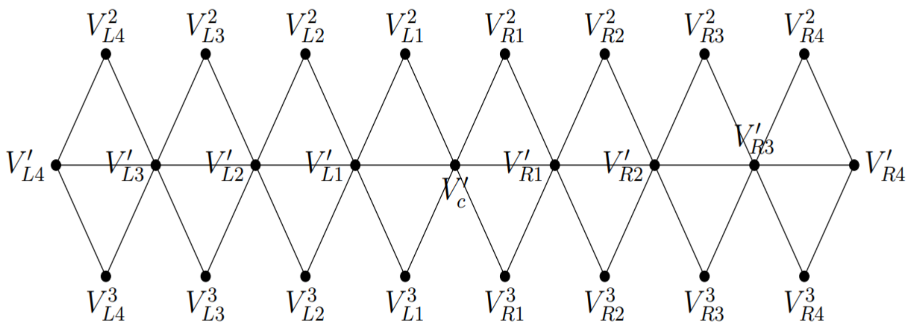

Figure 1:

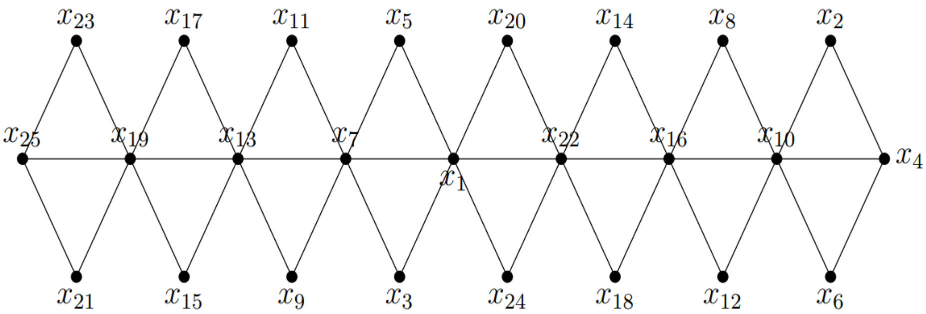

Figure 2

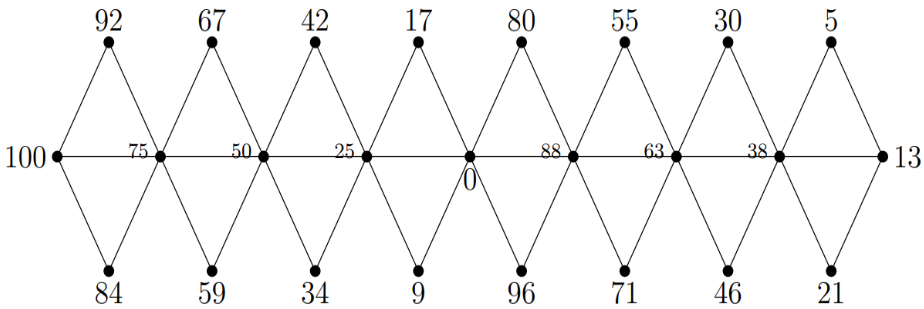

Figure 3

$rn(DT_9) = 100.$

Theorem 3.4. *Let* DT_n *be a Double Triangular snake graph on* n *vertices, Then* $rn(DT_n) = \frac{3n^2-6n+6}{2}$ *if n is even.*

Proof. Let h be an optimal radio labeling for DT_n and $\{x_1, x_2, \dots, x_p\}$ be the ordering of DT_n such that $0 = h(x_1) < h(x_2) < \dots < h(x_p)$. Then $h(x_{i+1}) - h(x_i) \geq (D+1) - d(x_i, x_{i+1})$ for all $1 \leq i \leq p-1$

Summing up these $p-1$ inequalities, we get

$$\sum_{i=1}^{p-1} [h(x_{i+1}) - h(x_i)] \geq \sum_{i=1}^{p-1} (D+1) - \sum_{i=1}^{p-1} d(x_i, x_{i+1})$$

$$h(x_p) - h(x_1) \geq (p-1)(D+1) - \sum_{i=1}^{p-1} d(x_i, x_{i+1})$$

$$h(x_p) \geq (p-1)(D+1) - \sum_{i=1}^{p-1} d(x_i, x_{i+1})$$

Therefore,

$$rn(DT_n) = h(x_p) \geq (p-1)(D+1) - \sum_{i=1}^{p-1} d(x_i, x_{i+1}) \quad (3.3)$$

Let $n = 2a$ and $a = \lfloor \frac{n}{2} \rfloor, a \geq 1$

In this case, diameter $D = 2a - 1$ and $p = 3n - 2$

From (2.2), we have $d(x_i, x_{i+1}) \leq L(x_i) + L(x_{i+1}) + 1, 1 \leq i \leq p-1$

$$\sum_{i=1}^{p-1} d(x_i, x_{i+1}) \leq \sum_{i=1}^{p-1} [L(x_i) + L(x_{i+1}) + 1]$$

$$\leq \sum_{i=1}^{p-1} [L(x_i) + L(x_{i+1})] + p - 1$$

$$= [L(x_1) + L(x_2) + \ldots + L(x_{p-1})] + [L(x_2) + L(x_3) + \ldots + L(x_p)] + p - 1$$

$$= \sum_{x \in V(DT_n)} L(x) - L(x_p) + \sum_{x \in V(DT_n)} L(x) - L(x_1) + p - 1$$

$$= 2 \sum_{x \in v(DT_n)} L(x) - L(x_1) - L(x_p) + p - 1$$

$$= 2L(DT_n) + p - 1 \text{ [choosing } x_1, x_p \in V(C(DT_n)), \; L(x_1) = L(x_p) = 0]$$

$$= \frac{3}{2} n(n-2) + 3n - 3 = \frac{3n^2-6}{2} \qquad (3.4)$$

substituting (3.4) in (3.3) we get

$$rn(DT_n) = h(x_p) \geq (3n-3)n - \left(\frac{3n^2-6}{2}\right)$$

$$= \frac{3n^2-6n+6}{2}$$

$$rn(DT_n) \geq \frac{3n^2-6n+6}{2}$$

Define a function $h: V(DT_n) \to \{0,1,2,\ldots,\frac{3n^2-6n+6}{2}\}$ by $h(x_1) = 0$ and $h(x_{i+1}) = h(x_i) + D + 1 - d(x_i, x_{i+1})$, for $1 \leq i \leq p-1$

Now, we label the vertices in the ordering as follows.

Let V_c^1, V_c^2, V_c^3 *and* V_c^4 be the central vertices of DT_n. We ordering the vertices of DT_n as follows. Let $v_{Li}^j, i = 1,2,\ldots,a, j = 1,2,3$ be the vertices on the left side with respect to the centres v_c^1, v_c^2 and v_c^3 while $v_{Ri}^j, i = 1,2,\ldots,a, j = 1,2,3$ be the vertices on the right side with respect to the centres $v_c^2, v_c^3 and v_c^4$ of DT_n.

Let $\{x_1, x_2, \ldots, x_p\}$ be the ordering of the vertices of DT_n. Label the vertices $x_1, x_2, \ldots, x_p$ as in the following procedure.

$$V_c^2 \to V_{R(a-1)}^1 \to V_{L1}^3 \to V_{R(a-1)}^2 \to V_{L1}^1 \to V_{R(a-1)}^3 \to V_{L1}^2$$

$$V_{R(a-2)}^1 \to V_{L2}^3 \to V_{R(a-2)}^2 \to V_{L2}^1 \to V_{R(a-2)}^3 \to V_{L2}^2$$

$$V_{R(a-3)}^1 \to V_{L3}^3 \to V_{R(a-3)}^2 \to V_{L3}^1 \to V_{R(a-3)}^3 \to V_{L3}^2$$

....

$$V_c^3 \to V_c^4 \to V_c^1$$

Thus, it is possible to assign labels to the vertices of DT_n with span equal to the lower bound satisfying the condition of lemma 3.1 and hence h is a radio labeling.

Thus, we have $rn(DT_n) \leq \frac{3n^2-6n+6}{2}$

Hence, $rn(DT_n) = \frac{3n^2-6n+6}{2}$

Example 3.5. *In Table 2, Figure 4, Figure 5 and Figure 6 an ordering of the vertices, renamed version, ordering version and optimal radio labeling for* DT_8 *are shown.*

$$V_c^2 \to V_{R3}^1 \to V_{L1}^3 \to V_{R3}^2 \to V_{L1}^1 \to V_{R3}^3 \to V_{L1}^2$$

$$V_{R2}^1 \to V_{L2}^3 \to V_{R2}^2 \to V_{L2}^1 \to V_{R2}^3 \to V_{L2}^2$$

$V_{R1}^{1} \to V_{L3}^{3} \to V_{R1}^{2} \to V_{L3}^{1} \to V_{R1}^{3} \to V_{L3}^{2}$
$V_{c}^{3} \to V_{c}^{4} \to V_{c}^{1}$

Table 2:

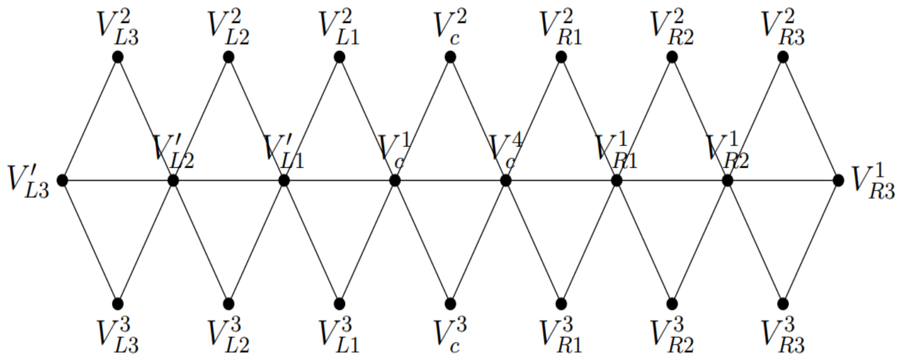

Figure 4:

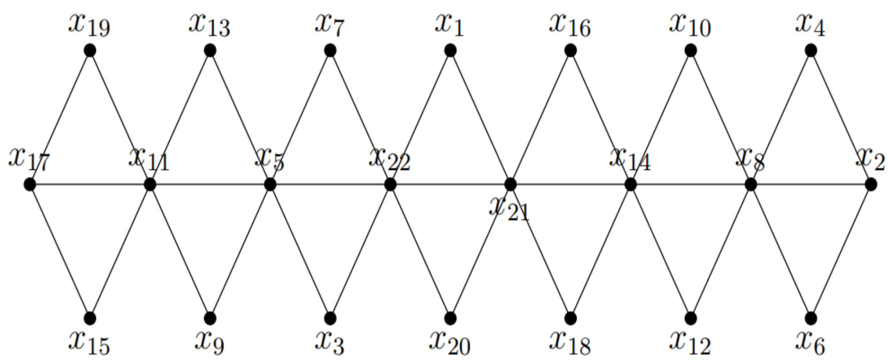

Figure 5:

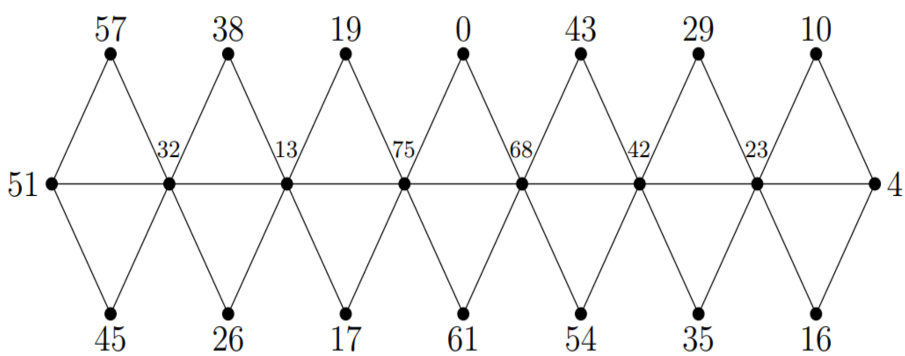

Figure 6:

$rn(DT_8) = 75.$

4. Conclusion

In this paper, we investigate the radio number of Double Triangular snake graphs. This can be extended to find the radio number of higher folds of Triangular snake graphs.

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Conflict of interest. As a sole author, there is no question of conflict of interest.

Authors' Contributions. The author did all the work for this paper.

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