On the Exponential Diophantine Equation $p.3^x + p^y = z^2$
with $p$ a Prime Number

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Received 6 August 2023; accepted 17 September 2023

Abstract. In this paper, we find non-negative integer solutions for exponential Diophantine equations of the type $p.3^x + p^y = z^2$ where $p$ is a prime number. We prove that such equation has a unique solution $(x,y,z) = (0,1,2)$ if $p = 2$. We also display the infinite solution set of that equation in the case $p = 3$. Finally, a brief discussion of the case $p ≡ 2 \pmod{3}$ is made, where we display an equation that does not have a non-negative integer solution and leave some open questions. The proofs are based on the use of the properties of the modular arithmetic.

Keywords: Congruences; exponential diophantine equations

AMS Mathematics Subject Classification (2010): 11A07, 11A41, 11D61

1. Introduction
Diophantine equations of the form $a^x + b^y = c^z$ have been studied by numerous mathematicians for many decades and by a variety of methods. One of the first references to these equations was given by Fermat-Euler [4], showing that $(a,c) = (5,3)$ is the unique positive integer solution of the equation $a^2 + 2 = c^4$. Several works on exponential Diophantine equations have been developed in recent years. In 2011, Suvarnamani [4] studied the Diophantine equation $2^x + p^y = z^2$. Rabago [5] studied the equations $3^x + 19^y = z^2$ and $3^x + 91^y = z^2$. The solution sets are $(1,0,2), (4,1,10)$ and $(1,0,2), (2,1,10)$, respectively. A. Suvarnamani et al. [7] found solutions of two Diophantine equations $4^x + 7^y = z^2$ and $4^x + 11^y = z^2$. Sroysang (see [6]) studied the Diophantine equation $3^x + 17^y = z^2$. Chotchaisthit (see [3]) showed that the Diophantine equation $p^x + (p + 1)^y = z^2$ has unique solutions $(p,x,y,z) = (7,0,1,3)$ and $(p,x,y,z) = (3,2,2,5)$ if $(x,y,z) \in \mathbb{N}^3$ and $p$ is a Mersenne prime. In 2019, Thongnak et al. (see [9]) found exactly two non-trivial solutions for the equation $2^x - 3^y = z^2$, namely $(1,0,1)$ and $(2,1,1)$. Buosi et al. (see [1] and [2]) studied some exponential Diophantine equations that generalized the work of Thongnak et al. (see [9]). Several other similar and recent works can be found in Thongnak et al. [10], [11] and [12].
In this work we show that when \( g_1^8 > 3 \) is a prime integer such that \( g_1^8 \equiv 2 \pmod{3} \), there is an ordered triple \((x, y, z)\) of non-negative integers that solves the equation \( p^x + p^y = z^2 \) if and only if \( p - 2 \) is a non-trivial power of 3. In the affirmative case, there exists only one solution which is given by \((x, y, z) = (\log_3(p - 2), 0, p - 1)\).

This result generalizes the theorem obtained in Thongnak et al. [10] when \( g_1^8 = 11 \). In other words, Thongnak et al. [10] found the only solution \((2, 0, 10)\) for the Diophantine equation \( 11^x + 11^y = 11^z \) using modular arithmetic. We also determine the unique solution of the case \( g_1^8 = 2 \) and the infinite set of solutions when \( g_1^8 = 3 \). The case where \( g_1^8 \) is congruent to 1 modulo 3 has not been solved completely because it is not understood why there are situations whose equation has a solution and others that do not. At the end of the article, a brief discussion of the case \( g_1^8 \equiv 1 \pmod{3} \) is made, showing an example of an equation with no solution and suggesting some open questions.

### 2. Some notations

Denote by \( \mathbb{Z} \) be the set of integer numbers and let \( \mathbb{N} \) be the set of all positive integers together with the number 0, that is, \( \mathbb{N} = \{0, 1, 2, 3, \ldots\} \), such a set will be called the set of **natural numbers**. Define \( \mathbb{N}^q = \mathbb{N} - \{0\} \) and \( \mathbb{N}^q = \mathbb{N} \times \mathbb{N} \times \cdots \times \mathbb{N} \) as the cartesian product of \( q \) copies of \( \mathbb{N} \). When \( a \) divides \( b \) we will use the symbol \( a \mid b \). When \( a \) is congruent to \( b \) module \( m \) we will write \( a \equiv b \pmod{m} \). Let \( a, m \) be integers with \( a > 0 \) and \( m > 2 \). The smallest positive integer \( k \) such that \( a^k \equiv 1 \pmod{m} \) will be said the **order** of \( a \) modulo \( m \) and will be denoted as \( |a|^m \). The set of all non-negative integer solutions of the equation \( p^x + p^y = z^2 \) will be said simply the **solution set of the equation**, i.e., the set \( \{(x, y, z) \in \mathbb{N}^3; p^x + p^y = z^2\} \).

### 3. Results

In this section, we will find the solution set for the equation
\[
p^x + p^y = z^2, \quad (x, y, z) \in \mathbb{N}^3,
\]
for several prime integers. We will divide the results into four sections: case \( p = 3 \), general results for \( p > 3 \), case \( p \equiv 2 \pmod{3} \) and finally, we will make a brief explanation of the case \( p \equiv 1 \pmod{3} \) since in this case the general problem still remains open. The motivation for this work is the paper Thongnak et al. [10] where the authors solved the above equation in the particular case \( p = 11 \). The result of Thongnak et al. is an immediate consequence of Theorem 1.13 proved in this article.

#### Case \( p = 3 \)

In this subsection, we present all the non-negative integer solutions of the equation \( p^x + p^y = z^2 \) in the particular case when \( p = 3 \).

**Theorem 1.1.** The solution set of the Diophantine exponential equation
\[
3 \cdot 3^x + 3^y = z^2
\]
in \( \mathbb{N}^3 \) is \( \{(2n, 2n, 2 \cdot 3^n); n \in \mathbb{N}\} \cup \{(1 + 2n, 3 + 2n, 2 \cdot 3^{n+1}); n \in \mathbb{N}\} \).

The proof is based on the combination of the results of the following six lemmas.

**Lemma 1.2.** If \((y, z) \in \mathbb{N}^2\) is a solution of the equation \(3^{y+1} + 1 = z^2\) then \( y = 0 \).

**Proof:** \(3^{y+1} + 1 = z^2 \Rightarrow 3^{y+1} = (z + 1)(z - 1) \Rightarrow z - 1 = 1, z + 1 = 3 \Rightarrow y = 0\). *Q.E.D.*
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Lemma 1.3. If \((x,y,z) \in \mathbb{N}^3\) is a solution of the equation \( 3^x(3^y+1) = z^2 \) then \( y = 0 \).

**Proof:** Suppose there exists \((x,y,z) \in \mathbb{N}^3\) such that \( 3^x(3^y+1) = z^2 \) and \( y > 0 \). By Lemma 1.2, \( 3^y+1 \) it is not a perfect square. Thus there is a prime integer \( q \) that appears an odd number of times in the prime factorization of \( 3^y+1 \). Since \( q \mid z^2 \) we have two possibilities:

- \( x = 0 \Rightarrow 3^y+1 \) is a perfect square;
- \( x > 0 \Rightarrow q \mid 3^y \Rightarrow q = 3 \Rightarrow 3 \mid 1 \).

In both cases we have an absurd. Therefore \( y = 0 \). \( Q.E.D. \)

Lemma 1.4. If \((x,y,z) \in \mathbb{N}^3\) is a solution of the equation (2) then \( y - x \in \{0,1,2\} \).

**Proof:** If \( y < x \) then there exists an integer \( k > 0 \) such that \( x = y + k \). Replacing \( x \) in (2) with \( y + k \) we obtain \( 3 \cdot 3^y + 3^y = z^2 \) if and only if \( 3^y(3^{y+1}+1) = z^2 \), which contradicts Lemma 1.3. Therefore \( y = x \in \{0,1,2\} \). \( Q.E.D. \)

Lemma 1.5. If \((x,y,z) \in \mathbb{N}^3\) is a solution of the equation (2) and \( y - x = 2 \) then there exists \( n \in \mathbb{N} \) such that \( x = 2n + 1 \) and \( y = 3n + 1 \).

**Proof:** Making \( y = x \) in equation (2) we get \( 3^{x+1} + 3^x = z^2 \Rightarrow 4 \cdot 3^x = z^2 \Rightarrow x \) is even.

Henceforth there exists \( n \in \mathbb{N} \) such that \( y = x = 2n \) and \( z = \sqrt{4 \cdot 3^{2n}} = 2 \cdot 3^n \). \( Q.E.D. \)

Lemma 1.6. If \((x,y,z) \in \mathbb{N}^3\) is a solution of the equation (2) and \( y - x = 2 \) then there exists \( n \in \mathbb{N} \) such that \( x = 1 + 2n \), \( y = 3 + 2n \) and \( z = 3^{n+1} \).

**Proof:** Making \( y = x + 2 \) in equation (2) we get \( 3^{x+1} + 3^{x+2} = z^2 \Rightarrow 4 \cdot 3^{x+1} = z^2 \Rightarrow x \) is odd.

Henceforth there exists \( n \in \mathbb{N} \) such that \( x = 1 + 2n \), \( y = 3 + 2n \) and \( z = \sqrt{4 \cdot 3^{2n+2}} = 2 \cdot 3^{n+1} \). \( Q.E.D. \)

**General results for a prime** \( p \neq 3 \)

Lemma 1.8. Let \( p \neq 3 \) be a prime integer. If \((x,y,z) \in \mathbb{N}^3\) is a solution of \( p^x + p^y = z^2 \),

then \( y = 0 \) or \( y = 1 \).
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**Proof:** Let \((x, y, z)\) be a solution of (3). Assume \(y \geq 2\). It is clear that \(z \neq 0\). In this case, \(p\) divides \(z\) because \(p3^x + p^y = z^2 \Rightarrow p(3^x + p^{y-1}) = z^2\).

Let \(m \in \mathbb{N}^*\) such that \(z = m \cdot p\). Substitute \(m \cdot p\) for \(z\) in the above equation to obtain \(3^x = p(m^2 - p^{y-2})\).

If \(x = 0\) the above equation is an absurd for all prime integer \(p \geq 2\). If \(x > 0\) we have \(3 \mid p\), which is absurd for all prime integer \(p \neq 3\). Therefore \(y = 0\) or \(y = 1\). \(Q.E.D.\)

If one substitute 0 for \(y\) in the equation (3) one obtain \(p3^x + 1 = z^2\) which is equivalent to the following equation
\[
p3^x = z^2 - 1 = (z - 1)(z + 1).
\]

**Lemma 1.9.** Let \(p \neq 3\) be a prime integer. If \((x, z) \in \mathbb{N}^2\) is a solution of (4), then \(x > 0\) and \(z\) it is not equivalent to 0 module 3.

**Proof:** Let \((x, z)\) be a solution of (4). If \(x = 0\) then \(p = (z - 1)(z + 1) \Rightarrow z = 2\) and \(p = 3\), which is a contradiction. Hence \(x > 0\). If 3 divides \(z\) then 3 divides \(z^2 - p3^x = 1\), which is an absurd. Therefore it is not equivalent to 0 module 3. \(Q.E.D.\)

We say that \(h\) is a non-trivial power of 3 if \(h = 3^x\) with \(x \in \mathbb{N}^*\).

**Lemma 1.10.** Let \(p > 3\) be a prime integer. The equation (4) has a solution in \(\mathbb{N}^2\) if and only if \(p - 2\) is a non-trivial power of 3 or \(p + 2\) is a non-trivial power of 3. In the affirmative case, the equation (4) has a unique solution in \(\mathbb{N}^2\) given by
\[
(\log_3(p - 2), p - 1) \text{ if } p - 2 \text{ is a non-trivial power of } 3;
\]
\[
(\log_3(p + 2), p + 1) \text{ if } p + 2 \text{ is a non-trivial power of } 3.
\]

**Proof:** Let \((x, z) \in \mathbb{N}^2\) be a solution of (4). By Lemma 1.9, \(x > 0\) and \(z\) it is not equivalent to 0 module 3.

If \(z \equiv 1(\text{mod } 3)\) then \(z - 1 \equiv 0(\text{mod } 3)\). Since \(p3^x = (z - 1)(z + 1)\) it follows that
\[
z + 1 = p \Rightarrow z = p - 1 \Rightarrow z = p - 1
\]
\[
z - 1 = 3^x \Rightarrow 3^x = z - 1 \Rightarrow p - 2 \Rightarrow x = \log_3(p - 2).
\]

If \(z \equiv 2(\text{mod } 3)\) then \(z + 1 \equiv 0(\text{mod } 3)\). Since \(p3^x = (z - 1)(z + 1)\) it follows that
\[
z - 1 = p \Rightarrow z = p + 1 \Rightarrow z = p + 1
\]
\[
z + 1 = 3^x \Rightarrow 3^x = z + 1 = p + 2 \Rightarrow x = \log_3(p + 2).
\]
The converse is straightforward and will be omitted. \(Q.E.D.\)

Making \(y = 1\) in the equation (3) one obtain
\[
p3^x + p = z^2.
\]

** Lemma 1.11.** Let \(p > 3\) be a prime integer. If \((x, y) \in \mathbb{N}^2\) is a solution of (5) then \(x > 0\) and \(z\) it is not equivalent to 0 module 3.

**Proof:** Suppose \((x, y) \in \mathbb{N}^2\) is a solution of (5). If \(x = 0\) then \(2p = z^2\), which is an absurd. It follows that \(x > 0\).
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If $z \equiv 0 \pmod{3}$ then $3$ divides $z^2 = p3^x + p$ and therefore $3$ divides $p$ which is a contradiction. \textit{Q.E.D.}

\textbf{Case} $p \equiv 2 \pmod{3}$

The Theorem 1.13 below presents all the non-negative integer solutions of the equation $p3^x + p^y = z^2$ in the particular case where $p \equiv 2 \pmod{3}$ and $p \neq 2$. This result generalizes Theorem 2.1 of [10] where $p = 11$.

\textbf{Lemma 1.12.} There is no $z \in \mathbb{Z}$ such that $z^2 \equiv 2 \pmod{3}$.

\textbf{Proof:} If $z \equiv 0 \pmod{3}$, then $z^2 \equiv 0 \pmod{3}$. If $z \equiv 1 \pmod{3}$ or $z \equiv 2 \pmod{3}$, then $z^2 \equiv 1 \pmod{3}$. \textit{Q.E.D.}

\textbf{Theorem 1.13.} Let $p > 3$ be a prime integer such that $p \equiv 2 \pmod{3}$. The equation

$$p3^x + p^y = z^2,$$

admits a solution in $\mathbb{N}$ if and only if $p - 2$ is a non-trivial power of $3$. In the affirmative case, the unique solution is $(x, y, z) = \left(\log_3(p - 2), 0, p - 1\right)$.

\textbf{Proof:} Let $(x, y, z)$ be a solution of (6). By Lemma 1.8 we must have $y = 0$ or $y = 1$. If $y = 0$, it follows from Lemma 1.10 that $(\log_3(p - 2), 0, p - 1)$ is the unique solution in $\mathbb{N}$ of the equation $p3^x + 1 = z^2$, since $\log_3(p - 2)$ is an integer. Now consider $y = 1$. By Lemma 1.11, $x \geq 1$. So we get $2 \equiv p \equiv z^2 - p3^x \equiv z^2 \pmod{3}$, which is a contradiction by Lemma 1.12. \textit{Q.E.D.}

\textbf{Remark 1.14.} For example, for $p = 17, 23, 41, 53, 59, 71$ the equation of the previous theorem has no non-negative integer solutions. For $p = 5, 11, 29, 83$ the solutions are respectively $(1, 0, 4), (2, 0, 10), (3, 0, 28)$ and $(4, 0, 82)$.

\textbf{Theorem 1.15.} The unique solution of the Diophantine exponential equation

$$2 \cdot 3^x + 2^y = z^2, (x, y, z) \in \mathbb{N}^2,$$

is the ordered triple $(x, y, z) = (0, 1, 2)$.

\textbf{Proof:} Let $(x, y, z)$ be a solution of (7). By Lemma 1.8 we must have $y = 0$ or $y = 1$. If $y = 0$, it follows from Lemma 1.9 that $x > 0$ and $z$ is not equivalent to $0$ modulo $3$. In this case we have the following equivalence for equation (7)

$$2 \cdot 3^x + 1 = z^2 \text{ if and only if } 2 \cdot 3^x = (z - 1)(z + 1) = z^2 - 1.$$  

If $z \equiv 1 \pmod{3}$ then $z - 1 \equiv 0 \pmod{3}$ and $z + 1 \equiv 2 \pmod{3}$, then we have

$$z + 1 = p \Rightarrow z = 1$$

$$z - 1 = 3^x \Rightarrow 3^x = 0'$$

an absurd.

If $z \equiv 2 \pmod{3}$ then $z - 1 \equiv 1 \pmod{3}$ and $z + 1 \equiv 0 \pmod{3}$, then we have

$$z + 1 = 3^x \Rightarrow z = 3$$

$$z - 1 = 2 \Rightarrow 3^x = 4'$$

an absurd.

Now consider $y = 1$. In this case equation (7) reduces to equation
2 \cdot 3^2 + 2 = 2(3^2 + 1) = z^2.

If \( x = 0 \) we have \( z^2 = 4 \), so \( z = 2 \). Therefore \((x, y, z) = (0, 1, 2)\) is a solution to equation (7) in \( \mathbb{N}^3 \). If \( x > 0 \) then \( z^2 \equiv 2(3^2 + 1) \equiv 2(\bmod 3) \), a contradiction by Lemma 1.12. Therefore \((x, y, z) = (0, 1, 2)\) is the only solution of equation (7). \textit{Q.E.D.}

4. When \( p \equiv 1(\bmod 3) \), the equation

\[ p^x + p^y = z^2, \quad (x, y, z) \in \mathbb{N}^3 \]  

has not yet been completely solved, that is, the behavior of the solutions of these equations is not known, whether they have a solution and whether the solutions, if any, are finite or infinite.

Let \((x, y, z)\) be a solution of (8). By Lemma 1.8, \( y \in \{0, 1\} \). By Lemma 1.10 we can say whether equation (8) will have a solution as long as \( p + 2 \) is a non-trivial power of 3. Furthermore, that lemma determines the unique solution in this case. However, for the case \( y = 1 \) we do not have a conclusive result for the time being. For example, equations with \( p = 7, 61 \) and 547 respectively have the following solutions \((3, 1, 14), (5, 1, 122)\) and \((7, 1, 1094)\). We do not know if those three equations have other solutions.

\textbf{Remark 1.16.} Note that \((q, 2) \in \mathbb{N}^2\) is a solution of \( 3^x + 1 = p \cdot w^2 \) if and only if \((q, 1, 2p)\) is a solution of \( p^x + p^y = z^2, (x, y, z) \in \mathbb{N}^3 \).

In the next theorem we will show an example whose given equation does not have non-negative integer solutions.

\textbf{Theorem 1.17.} The exponential Diophantine equation

\[ 13 \cdot 3^x + 13^y = z^2, \quad (x, y, z) \in \mathbb{N}^3, \]  

has no solutions.

\textbf{Proof:} Let \((x, y, z)\) be a solution of (9). By Lemma 1.8, \( y \in \{0, 1\} \). First consider \( y = 0 \). By Lemma 1.10 there are no solutions to the equation in this case, since \( p + 2 = 15 \) is not a non-trivial power of 3.

Suppose there is a solution \((x, 1, z) \in \mathbb{N}^3\) of (9). In this case equation (9) reduces to \( 13 \cdot 3^x + 13 = z^2 \). Note that 13 divides \( z \) and therefore \( z = 13w, w \in \mathbb{N}^* \). So we have the following equivalence of equations

\[ 13 \cdot 3^x + 13 = z^2 = 13^2 \cdot w^2 \text{if and only if } 3^x + 1 = 13 \cdot w^2 \equiv 0(\bmod 13). \]

On the other hand, notice that \( 3^2 \equiv 9(\bmod 13) \) and \( 3^3 \equiv 1(\bmod 13) \). Therefore the order of 3 modulo 13 is equal to 3, that is \( [3]_{13} = 3 \). So write \( x = 3m + r \), where \( m \in \mathbb{N} \) and \( r \in \{0, 1, 2\} \). So we have the following equation

\[ 3^x + 1 = 3^{3m+r} + 1 = 27^m \cdot 3^r + 1 \equiv 3^r + 1(\bmod 13) \equiv 4(\bmod 13), \text{ if } r = 1, \]

\[ 10(\bmod 13) \quad r = 2 \]

an absurd. \textit{Q.E.D.}

5. \textbf{Open questions}

The following questions refer to the equation

\[ p^x + p^y = z^2, (x, y, z) \in \mathbb{N}^3 \text{ with } p \equiv 1(\bmod 3). \]  

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- When $p \equiv 1 \pmod{3}$, what additional conditions must exist on $p$ for the equation (11) to have a solution?
- If there is a solution for equation (10), how do you know if the number of solutions is finite or infinite?
- What additional conditions must be imposed on $p$ for there to be a unique solution?

Acknowledgements. We thank the reviewers for their careful reading of our manuscript and the useful comments.

Conflicts of Interest: The authors declare that there is no conflict of interest.

Author’s Contributions: All authors contributed equally.

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