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A Ramsey Problem Related to Butterfly Graph vs. Small Paths and C_3

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Abstract. A graph on five vertices consisting of 2 copies of the cycle graph C_3 sharing a common vertex is called the Butterfly graph (B). The smallest natural number s such that any two-colouring (say red and blue) of the edges of $K_{j\times s}$ has a copy of a red B or a blue G is called the multipartite Ramsey number of Butterfly graph versus G. This number is denoted by $m_j(B,G)$. In this paper, we find the exact values for $m_j(B,G)$ when $j>2$ and G represents any small path or else three cycles.

Keywords: Graph theory, Ramsey theory, Ramsey critical graphs

AMS Mathematics Subject Classification (2010): 05C55, 05C38, 05D10

1. Introduction

In this paper, we concentrate on simple graphs. Let the complete multipartite graph having j uniform sets of size s be denoted by $K_{j\times s}$. Given two graphs G and H, we say that $K_N \rightarrow (G, H)$ if K_N is coloured by two colours, red and blue, and it contains a copy of G (in the first color red) or a copy of H (in the second color blue). Regarding this notation, we define the Ramsey number r(n,m) as the smallest integer N, such as $K_N \rightarrow (K_n, K_m)$. As of today, beyond the case $n = 5$, almost nothing significant is known with regard to diagonal classical Ramsey number $r(n,n)$ (see [8] for a survey). Burger and Vuuren (see [1]) were honoured for introducing and developing a branch of Ramsey numbers known as size multipartite Ramsey numbers. The size multipartite Ramsey number $m_j(B,G)$, which is a generalization of the much celebrated Ramsey number, is based on exploring the two colourings of multipartite graph $K_{j\times s}$ instead of the complete graph. Formally, we define size multipartite Ramsey number as the smallest natural number s such that $K_{j\times s} \rightarrow (K_n, K_m)$.

In the last 14 years, many research papers have been published on the Ramsey number for different pairs of graphs. [2,3,4,6]. Works of [5,7], focuses on the multipartite Ramsey numbers for graph G versus graph H where H is any isolated vertex free simple graph on four vertices and graph G refers to either a C_3, a C_4. In this paper we find exact the values for $m_j(B,G)$ when $j>2$ and G represents any small path or else a 3 cycle.

$G =$	P_2	P_3	P_4	C_3
$j=3$	2	2	3	∞
$j=4$	2	2	2	∞
$j=5$	1	1	2	∞
$j=6$	1	1	2	2
$j=7$	1	1	1	2
$j=8$	1	1	1	2
$j\geq 9$	1	1	1	1

2. Notation

Given a graph $G=G(V,E)$ the *order* of the graph is denoted by $|V(G)|$ and the *size* of the graph is denoted by $|E(G)|$. For a vertex v of a graph G, the neighborhood of v, denoted by $N(v)$ is defined as the set of vertices adjacent to v. Furthermore, the cardinality of this set, denoted $d(v)$, is defined as the degree of v. In a Butterfly graph B, the vertex of degree 4 is defined as the center of the Butterfly graph B. We say that a graph G is a k regular graph if $d(v) = k$ for all $v\in V(G)$. Let $N_R(v)$ ($N_B(v)$) be the set of vertices adjacent to v in red(blue). Then the cardinality of this set is denoted by $deg_R(v)$ ($deg_B(v)$). Denote the j partite sets of $K_{j\times s}$ by V_1, V_2, ... ,V_j. Let $K_{j\times s} = H_R \oplus H_B$ denote a red and blue coloring of $K_{j\times s}$ where H_R consists of the red graph and where H_B consists of the blue graph, having vertex sets equal to $V(K_{j\times s})$. Suppose that a vertex $u \in V(K_{j\times s})$ of H_R (or H_B) belonging to the partite set V_i is such that it is incident to $i_1,i_2,...,i_{j-1}$ vertices of each of the remaining j-1 partite sets respectively. Then, we say that vertex u has a $(i_1,i_2,...,i_{j-1})$ red (or blue) split in H_R (or H_B) provided that $i_1 \geq i_2 \geq i_3 \geq \ldots \geq i_{j-1}$. Moreover if there exists $k_1,k_2,...,k_{j-1}$ such that $i_1 \geq k_1$, $i_2 \geq k_2$, $i_3 \geq k_3, \ldots\ i_{j-1} \geq k_{j-1}$ and $k_1 \geq k_2 \geq k_3 \geq \ldots \geq k_{j-1}$, then we say that u contains a $(k_1,k_2,...,k_{j-1})$ red (or blue) split in H_R (or H_B).

3. Size Ramsey numbers for $m_j(B,P_2)$ and $m_j(B,P_3)$

Theorem 3.1. *If* $j \geq 3$, then

$$m_j(B,P_2) = \begin{cases} 2 & \text{if } j \in \{3,4\} \\ 1 & \textit{otherwise} \end{cases}$$

Proof of Theorem 3.1: The proof is trivial and is left for the reader.

Theorem 3.2. If $j \geq 3$, then

$$m_j(B,P_2) = \begin{cases} 2 & \text{if } j \in \{3,4\} \\ 1 & \textit{otherwise} \end{cases}$$

Proof of Theorem 3.2: Consider the red-blue coloring of $K_{3\times 2} = H_R \oplus H_B$ where H_B consists of three independent blue edges $(v_{1,1},v_{2,2})$, $(v_{2,1},v_{3,2})$ and $(v_{1,2},v_{3,1})$. Then H_R will consist of the following diagram. Thus, $K_{3\times 2}$ has neither a blue P_3 nor a red B. Therefore, $m_3(B,P_3) \geq 3$.

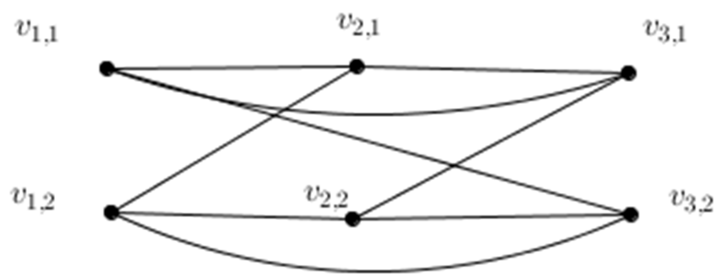

Figure 3.1: The H_R graph

Next to show $m_3(B,P_3) \leq 3$, consider any red/blue coloring given by $K_{3\times 3} = H_R \oplus H_B$ such that H_R contains no red B and H_B contains no blue P_3. In order to avoid a blue P_3 all vertices must have blue degree at most equal to 1. That is all vertices must have red degree at least equal to 5. Since $K_{3\times 3}$ has odd number of vertices without loss of generality, we may assume that $v_{1,1}$ has red degree 6. However, as $deg_R(v_{2,1}) \geq 5$ and $deg_R(v_{2,2}) \geq 5$, there will be a red $2K_2$ induced by $V_2 \cup V_3$. Thus, we will get a red B, a contradiction. Therefore, $m_3(B,P_3) = 3$.

As, $r(B,P_3) = 5$, (see [4]) we get $m_4(B,P_3) \geq 2$. Next to show, $m_4(B,P_3) \leq 2$, consider any red/blue coloring given by $K_{4\times 2} = H_R \oplus H_B$, such that H_R contains no red B and H_B contains no blue P_3. In order to avoid a blue P_3 all vertices must have blue degree at most equal to 1. That is all vertices must have red degree at least equal to 5. Suppose that $v_{1,1}$ is adjacent in red to all vertices of $U=\{v_{2,1}, v_{2,2}, v_{3,1}, v_{3,2}, v_{4,1}\}$. But then in order to avoid a red B induced by$\{v_{1,1}, v_{2,1}, v_{2,2}, v_{3,1}, v_{3,2}, v_{4,1}\}$, U must not contain a red $2K_2$. That is, U must contain a blue $K_{1,2}$, a contradiction. Therefore, $m_4(B,P_3) = 2$.

As, $r(B,P_3) = 5$, we get $m_j(B, P_3) = 1$ for $j \geq 5$.

Theorem 3.3. If $j \geq 3$, then

$$m_j(B,C_3) = \begin{cases} \infty & j \in \{3,4,5\} \\ 2 & j \in \{6,7,8\} \\ 1 & otherwise \end{cases}$$

Proof of Theorem 3.3: $(B,C_3) = \infty$ since $m_j(C_3,C_3) = \infty$ for $j \in \{3,4,5\}$ and C_3 is a subgraph of B (See [5]).

Next consider the case $j \in \{6,7,8\}$. First consider a red/blue coloring of $K_{6\times 2}$, given by $K_{6\times 2} = H_R \oplus H_B$, such that H_R contains no red B and H_B contains no blue C_3. As $m_6(C_3,C_3) = 1$, the induced subgraph H_1 where $V(H_1) = \{v_{i,1} : i \in \{1,2,...,6\}\}$ has a red C_3 say $v_{1,1},v_{2,1},v_{3,1},v_{1,1}$. Denote this red C_3 by A_1. Similarly the induced subgraph H_2 where $V(H_2)$

= $\{v_{3,1}\} \cup \{v_{i,2} : i \in \{1,2,4,5,6\}\}$ has a red C_3 (say A_2). If v_{31} is a vertex of A_2 then $K_{6\times2}$ has a red B, a contradiction. Otherwise, we get the following three cases.

Case 1: None of the vertices of A_2 belong to the partite sets V_1,V_2,V_3.

Case 2*:* Two of the vertices of A_2 belong to two of the partite sets V_1,V_2,V_3.

Case 3: Only one of the vertices of A_2 belong to one of the partite sets V_1,V_2,V_3.

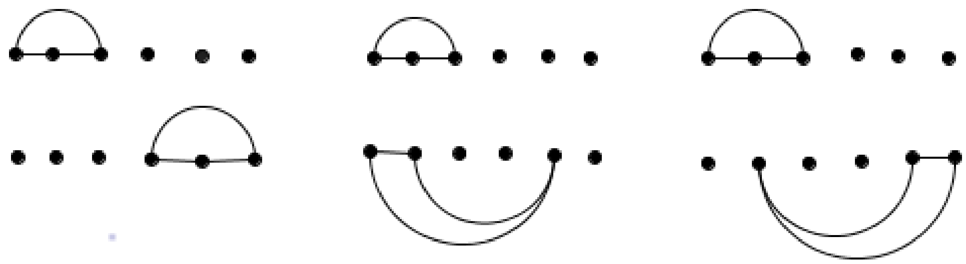

Figure 3.2: The three cases

In each of these cases first consider the induced subgraph H such that H consists of six vertices $v_1,v_2,...,v_6$ where no two vertices of $\{v_1,v_2,...,v_6\}$ belong to the same partite set and $|V(A_i) \cap V(H)| = 1$ for each $i \in \{1,2\}$. Due to the absence of a blue C_3 and $m_6(C_3,C_3) = 1$, H has a red C_3 (say A_3). If $V(A_3) \cap V(A_1) \neq \emptyset$ or $V(A_3) \cap V(A_2) \neq \emptyset$ then $K_{6\times2}$ has a red B, a contradiction.

Otherwise, consider the induced subgraph H_1 consisting of the six vertices $u_1,u_2, ... ,u_6$ where no two vertices of $\{u_1,u_2, ...,u_6\}$ belong to the same partite set and $|V(A_i) \cap V(H_1)| = 2$ for each $i \in \{1,2,3\}$. H_1 has a red C_3 due to the absence of a blue C_3 and $m_6(C_3,C_3) = 1$. This red C_3 along with one of the A_i where $i \in \{1,2,3\}$ forms a red B, a contradiction. Therefore, $m_6(B,C_3) \leq 2$.

As, $r(B,C_3) =9$, (see [4]) we get, $m_8(B,C_3) \geq 2$.

Therefore, $2 \leq m_8(B,C_3) \leq m_7(B,C_3) \leq m_6(B,C_3) \leq 2$, gives us $m_j(B,C_3) = 2$ for $j \in \{6,7,8\}$. Finally, as $r(B,C_3) =9$, (see [4]) we get, $m_j(B,C_3) =1$ if $j \geq 9$. □

Theorem 3.4. *If* $j \geq 3$, *then*

$$m_j(B,P_4) = \begin{cases} 3 & j=3 \\ 2 & j \in \{4,5,6\} \\ 1 & otherwise \end{cases}$$

Proof of Theorem 3.4: $m_3(B,P_4) \geq 3$. since $m_3(B,P_3) = 3$ by theorem 2.

Next to show, $m_3(B,P_4) \leq 3$, consider any red/blue coloring of $K_{3\times3}$ given by $K_{3\times3} = H_R \oplus H_B$, such that H_R contains no red B and H_B contains no blue P_4. By the theorem 2 as $m_3(B,P_3)$=3, we get that there exists a blue P_3.

Case 1: There exists a blue P_3 that lies in three partite sets.

Without loss of generality, assume that this blue P_3 comprises of $(v_{1,1},v_{2,1})$ and $(v_{2,1},v_{3,1})$ blue edges. But then in order for $K_{3\times3}$ not to have a blue P_4, $(v_{3,1},v_{1,2})$, $(v_{3,1},v_{1,3})$, $(v_{3,1},v_{2,2})$ and $(v_{3,1},v_{2,3})$ have to be red edges. Next for $W=\{v_{1,2},v_{1,3},v_{2,2},v_{2,3},v_{3,1}\}$ not to induce a red B, W will be forced to contain a blue P_3, belonging to V_1 and V_2. Thus, this case leads to the following case 2.

Case 2: There exists a blue P_3 that lies in two partite sets.

Without loss of generality, assume that this blue P_3 comprises of $(v_{1,1},v_{2,1})$ and ($v_{1,1}$, $v_{2,2}$) blue edges. But then in order for $K_{3\times3}$ not to have a blue P_4, $\{v_{2,1},v_{2,2}\}$ will have to be adjacent to all vertices of $W_1=\{v_{1,2},v_{1,3},v_{3,1},v_{3,2},v_{3,3}\}$ in red. In order for W_1 not to induce a blue P_4, without loss of generality we may assume that, $(v_{1,2},v_{3,1})$ is a red edge. But then in order to avoid a red B induced by W_1U$\{v_{2,1}\}$, $(v_{1,3}, v_{3,2})$ and $(v_{1,3}, v_{3,3})$ are blue edges. In addition, in order to avoid blue P_4, given by $v_{3,2}$ $v_{1,3}$ $v_{3,3}$, $v_{1,2}$ the edge $(v_{3,3}, v_{1,2})$ are a red edge. But then $\{v_{1,2},v_{2,1},v_{2,2},v_{3,1},v_{3,3}\}$ will induce a red B consisting of the two red triangle $v_{1,2},v_{2,2},v_{3,1},v_{1,2}$ and $v_{1,2},v_{2,1},v_{3,3},v_{1,2}$ with $v_{1,2}$ as the center vertex, a contradiction.

Thus, $m_3(B,P_4) \leq 3$. Therefore, $m_3(B,P_4) = 3$.

As $r(B,P_4) = 7$, (see [4]) we get $m_6(B, P_4) \geq 2$. To show $m_4(B, P_4) \leq 2$, consider $K_{4\times2}$ with any red/blue coloring. Assume $K_{4\times2}$ has neither a red B nor a blue P_4. Since $m_4(B,P_3) = 2$ and $K_{4\times2}$ has no red B, it has a blue P_3.

Case 1: There exists a blue P_3 that lies in two partite sets

Let the blue P_3 be $v_{1,1}$, $v_{2,1}$, $v_{1,2}$. As there is no blue P_4 all vertices in $V_3 \cup V_4 \cup \{ v_{2,2}\}$ are adjacent in red to both $v_{1,1}$ and $v_{1,2}$. As there is no red B the red graph induced by $H=\{ v_{2,2}, v_{3,1}, v_{3,2}, v_{4,1}, v_{4,2} \}$ has no red $2K_2$. Then any connected components in the graph induced by H is equal to a K_1,, K_2, P_3, K_3, $K_{1,3}$ or $K_{1,4}$. Also the induced red graph of H can contain at most one connected component having one or more red edges. In both these situations the blue graph induced by H has a blue P_4, a contradiction.

Case 2: There exists a blue P_3 that lies in three partite sets.

Let the blue P_3 be $v_{1,1}$, $v_{2,1}$, $v_{3,1}$. As there is no blue P_4 all vertices in $\{v_{i,2}: i\in \{2,3,4\}\} \cup \{v_{4,1}\}$ are adjacent in red to $v_{1,1}$ and all vertices in $\{v_{i,2} : i \in \{1,2,4\}\} \cup \{v_{4,1}\}$ are adjacent in red to $v_{3,1}$. However, by the elimination of case 1, either $(v_{3,2},v_{4,1})$ or $(v_{3,2},v_{4,2})$ must be red. Without loss of generality assume that $(v_{3,2} ,v_{4,1})$ is red. In order to avoid a red B with $v_{1,1}$ as the center, $(v_{2,2},v_{4,2})$ must be blue. But then, in order to avoid case 1, $(v_{2,2},v_{4,1})$ is red. Next in order to avoid a red B with $v_{1,1}$ as the center, $(v_{3,2},v_{4,2})$ must be blue and in order to avoid a red B with $v_{3,1}$ as the center, $(v_{1,2},v_{4,2})$ must be blue. However, as there is no blue P_4, $(v_{1,2},v_{2,2})$, $(v_{2,2},v_{3,2})$ and $(v_{1,2},v_{3,2})$ must all be red. This gives us a red B with $v_{2,2}$ as the center, a contradiction. Therefore, $m_4(B,P_4) \leq 2$.

As $2 \leq m_6(B, P_4) \leq m_5(B, P_4) \leq m_4(B, P_4) \leq 2$ we get $m_j(B, P_4) = 2$ for $j \in \{4,5.6\}$. Finally, as $r(B,P_4) = 7$ we get, $m_j(B, P_4) = 1$ if $j \geq 7$.

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REFERENCES

1. P. Burger and J. H. van Vuuren, Ramsey numbers in Complete Balanced Multipartite Graphs. Part II: Size Numbers, *Discrete Mathematics*, 283 (2004) 45-49.
2. P. Erdös, R. J. Faudree, C. C. Rousseau and R. H. Schelp, The size Ramsey number, *Period Mathematics Hungary*, 9 (1978) 145-161.
3. R. J. Faudree and R. H. Schelp, Path-path Ramsey-type numbers for the complete bipartite graph, *Journal of Combinatorial Theory (B)*, 19 (1975) 161-173.
4. G. R. T. Hendry, Ramsey Numbers for Graphs with Five Vertices, *Journal of Graph Theory*, 13 (1989) 245-248.
5. C.J. Jayawardene and B. L. Samarasekara, Size multipartite Ramsey numbers for C_3 versus all graphs on 4 vertices, *Journal of National Science Foundation, Sri Lanka*, 45 (2017) 67-72.
6. C.J. Jayawardene and C. C. Rousseau, Ramsey numbers $r(C_6,G)$ for all graphs of order less than six, *Congressus Numerantium (*1999) 147-160.
7. C.J.Jayawardene and T.U.Hewage, Multipartite Ramsey numbers for C_4 versus all graphs on 4 vertices, *Annals of Pure and Applied Mathematics*, 13(2) (2017) 297-304.
8. V.Kavitha and R.Govindarajan, A study on Ramsey numbers and its bounds, *Annals of Pure and Applied Mathematics*, 8(2) (2014) 227-236.