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# More on the Exponential Diophantine Equation $23^x + 233^y = z^2$

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***Abstract.*** In this paper, it is shown that the exponential diophantine equation $23^x + 233^y = z^2$ is found to have a unique solution $(x, y, z) = (1, 1, 16)$ in non-negative integers x, y, and z by using Catalan's conjecture, factorization methods, and modular arithmetic, and elementary mathematical concepts. Moreover, its generalization is proved at the end.

***Keywords:*** Exponential diophantine equation, Catalan's conjecture, integer solution, modular arithmetic, non-linear equation

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## 1. Introduction

In 1844 Eugene Charles Catalan conjectured a theorem known as Catalan's Conjecture [1] (or Mihailescu's theorem [2]). It was proved by Mihailescu in 2002 at Paderborn University. In 2007, Dumitru [3] proved that the Diophantine equation $2^x + 5^y = z^2$ has exactly two non-negative integer solutions $(x, y, z) = (3,0,3), (2,1,3)$. Nechemia Burshtein [6, 8, 11, 12, 14] worked on the solutions of the Diophantine equations of the form $p^x + q^y = z^2$. In 2018, Rao [7] proved that the Diophantine equation $3^x + 7^y = z^2$ has exactly two solutions in non-negative integers $(x, y, z) = (1, 0, 2)$ and $(2, 1, 4)$. In 2019, Asthana and Singh [9] proved that the Diophantine equation $53^x + 143^y = z^2$ has exactly two solutions $(x, y, z) = (0,1,12), (1,1,14)$. In 2019, Burshtein [8] proved that the Diophantine Equation $11^x + 23^y = z^2$ with Consecutive positive integers x, y, has exactly one solution $(x, y, z) = (2,1,2)$. In 2013, Rabago [4] worked on two diophantine equations $3^x + 19^y = z^2$ and $3^x + 91^y = z^2$. In 2013 Chotchaisthit [5] solved the Diophantine equation $2^x + 11^y = z^2$. In 2023, Srimud and Tadee [10] worked on the Diophantine equation $3^x + b^y = z^2$, where b is a positive integer such that $b \equiv 5 \bmod 20$ or $b \equiv 5 \bmod 30$ for non-negative integer solutions.

In 2023, Orosram [13] worked on the Diophantine equation $(p+n)^x + p^y = z^2$, where $p, p+n$ are prime numbers and n is a positive integer such that $n \equiv 0 \pmod 4$. But for $p = 23, n = 210, p+n = 233$ where $n \equiv 2 \pmod 4$, this paper finds the gap in the recent work by Orosram [13].

In 2020, Burstein [12] worked on the Diophantine equation $(10K + A)^x + (10M + A)^y = z^2$, for $A = 1,3,7,9$. In which he proved that it has infinitely many integer solutions for $A = 3$, and $K = 10M^2 + 7M + 1$, K, M are integers. But for $K = 2$ and $A = 3$, $M = 23$ and as $K \neq 10M^2 + 7M + 1$ this paper finds the gap in the work by Burshtein[12].

Thus an attempt is made to solve the Diophantine equation $p^x + q^y = z^2$ with the prime numbers $\mathrm{p} = 23, \mathrm{q} = 233$ and $\mathrm{q} - \mathrm{p} = 210$.It has a unique solution in non-negative integers (x, y, z) = (1, 1, 16). Hence as a generalization the Diophantine equation $23^{\mathrm{x}} + 233^{\mathrm{y}} = \mathrm{w}^{kn}$ is investigated for non-negative integers in $x, y, w, k > 0, n > 0$ and $kn > 1$, $kn$ is an even positive integer also.

## 2. Preliminaries

**Proposition 2.1** (Catalan's Conjecture)

The only solution of the Diophantine equation $\mathrm{a}^{\mathrm{x}} - \mathrm{b}^{\mathrm{y}} = 1$ is (a, x, b, y) = (3, 2, 2, 3), where a, b, x and y are integers with minimum $\{\mathrm{a}, \mathrm{b}, \mathrm{x}, \mathrm{y}\} > 1$.

**Proof:** This conjecture was proved by Mihailescu [2] in 2004.

Now we will prove the following three lemmas by the Catalan's conjecture.

**Lemma 2.1.** Let x and z be non-negative integers. The Diophantine equation $23^{\mathrm{x}} + 1 = \mathrm{z}^2$ has no solutions.

**Proof:** let x and z be non-negative integers and $23^{\mathrm{x}} + 1 = \mathrm{z}^2$ (1)

If $x = 0$ then, $\mathrm{z}^2 = 2$, this not solvable for integers.

If $z = 0$ then $23^{\mathrm{x}} = -1$, this is impossible.

If $x = 1$ then $\mathrm{z}^2 = 24$, no integer solution

When $z = 1$ there is impossibility. So let $x > 1$ *and* $z > 1$.

Then clearly $\min\{x, y, 23, 2\} > 1$ and from (1), we get $23^{\mathrm{x}} - \mathrm{z}^2 = 1$.

By Catalan's Conjecture 2.1, the equation (1) has no solutions.

**Lemma 2.2.** The Exponential Diophantine equation $1 + 233^{\mathrm{y}} = \mathrm{z}^2$ has no non-negative integer solution in y and z.

**Proof:** If possible let y and z be non-negative integers such that $233^y + 1 = z^2$. (2)

If y=0, then $z^2 = 2$ which is not solvable.

If z=0 then $233^y = -1$ which is impossible.

If $y = 1$ then $z^2 = 234$ this is not solvable.

The case $z = 1$ will never occur.So take $y > 1, z > 1$. Then clearly

$\min\{x, y, 233,2\} > 1$.

By Catalan's Conjecture 2.1, the equation (2) has no solutions.

**Lemma 2.3.** Suppose $x, y, z$ are non-negative integers related by the Exponential Diophantine equation $23^x + 233^y = z^2$. Then $z$ is even only if and only if either $x$ is odd and $y$ is even or $x$ is even and $y$ is odd only.

**Proof:** Let $x, y, z$ be non-negative integers such that $23^x + 233^y = z^2$.

We know that $23^x \equiv \begin{cases} 3 \bmod 4 \text{ if } x \text{ is odd} \\ 1 \bmod 4 \text{ if } x \text{ is even} \end{cases}$

and $233^y \equiv \begin{cases} 3 \bmod 4 \text{ if } y \text{ is odd} \\ 1 \bmod 4 \text{ if } y \text{ is even} \end{cases}$

**Case 1.** When both $x$ and $y$ are even or both $x$ and $y$ are odd.

We get $z^2 = 23^x + 233^y \equiv 2 \bmod 4$

This is a contradiction to $z^2 \equiv 0 \bmod 4 \text{ or } 1 \bmod 4$

Therefore there is no solution when both $x$ and $y$ are even or odd.

**Case 2.** Either $x$ is odd and $y$ is even or $x$ is even and $y$ is odd.

Then we get $z^2 = 23^x + 233^y \equiv 0 \bmod 4$. Hence $z$ is even only.

## 3. Main results

**Theorem 3.1.** Let x, y and z be non-negative integers. Then the exponential Diophantine equation $23^x + 233^y = z^2$ has the unique solution $(x, y, z) = (1, 1, 16)$.

**Proof:** Let x, y and z be non-negative integers such that $23^x + 233^y = z^2$. (3)

**Case 1.** When $y = 0$

By the lemma 2.1 the equation (3) has no solutions.

**Case 2.** When $x = 0$

By the Lemma 2.2 the equation (3) has no solutions.

**Case 3.** When $x \geq 1$ and $z \geq 1$.

In view of lemma 2.3, it is enough to consider two cases only to prove the theorem.

**Subcase 3.1.** Suppose that $y$ is odd and x is even.

When y is odd, i.e. let $y = 2k + 1$. Here $k$ is a non-negative integer.

We will separate this case into two parts: Part I and Part II.

**Part I.** $23^x + 233^{2k+1} = z^2$ or $23^x + (8 + 225)233^{2k} = z^2$

i.e. $23^x + 8.233^{2k} = z^2 - 225.233^{2k}$=$\left(z - 15.233^k\right)\left(z + 15.233^k\right)$

There are two possibilities for this equation

$$\begin{cases} z - (15.233^k) = 1 \\ z + (15.233^k) = 23^x + (8.233^{2k}) \end{cases} \quad \text{or} \quad \begin{cases} z + (15.233^k) = 1 \\ z - (15.233^k) = 23^x + (8.233^{2k}) \end{cases}$$

Solving the first set of equations we get $\left(30.233^k\right) = 23^x + (8.233^{2k}) - 1$

$23^x - 1 = \left(30.233^k\right) - \left(8.233^{2k}\right) = 233^k\left(30 - \left(8.233^k\right)\right)$

Then $233^k = 1$ and $\left(30 - \left(8.233^k\right)\right) = 23^x - 1$

$\Longrightarrow k = 0$ , $23^x = 23 \Longrightarrow x = 1$. So that $y = 1$. $z^2 = 256. \Longrightarrow z = 16$

Thus there is a solution $(x, y, z) = (1,1,16)$ (4)

Solving the second set of equations, $\left(30.233^k\right) = 1 - 23^x - \left(8.233^{2k}\right)$

$\Longrightarrow 1 - 23^x = 8.233^{2k} + \left(30.233^k\right)$= $233^k(30 + \left(8.233^k\right))$

$\Longrightarrow$ k=0 and $23^x = -37$ this is impossible.

**Part II.** Again we have $23^x + 233^{2k+1} = z^2$ or $23^x + (256 - 23)233^{2k} = z^2$ .

So that $23^x + (-23)233^{2k} = z^2 - (256)(233^{2k})$

$$=\left(z - (16)(233^k)\right)\left(z + (16)(233^k)\right)$$

There are two possibilities for this equation

$$\begin{cases} \left(z - (16)(233^k)\right) = 1 \\ z + (16)(233^k) = 23^x + (-23)233^{2k} \end{cases} \quad \text{Or} \quad \begin{cases} z - (16)(233^k) = 23^x + (-23)233^{2k} \\ z + (16)(233^k) = 1 \end{cases}$$

Solving the first set of equations, $23^x - 1 = 233^k\left(32 + 23.233^k\right)$

$\Longrightarrow k = 0$ and $23^x = 56$. This is not solvable for $x$.

Solving the second set of equations, $23^x - 1 = 233^k(23.233^k - 32)$

$\Longrightarrow k = 0$ and $23^x = -8$, this is impossible.

**Subcase 3.2.** Suppose that $x$ is odd and y is even

When $x$ is odd i.e., $x = 2k + 1$ for some non-negative integer $k$, then

$$233^y = z^2 - 23^{2k+1} = z^2 - (16+7)23^{2k}$$

$$\Longrightarrow 233^y + (7)23^{2k} = z^2 - (16)23^{2k} = \left(z - (4)23^k\right)\left(z + (4)23^k\right).$$

There are two possibilities for this equation

$$\begin{cases} \left(z - (4)23^k\right) = 1 \\ \left(z + (4)23^k\right) = 233^y + (7)23^{2k} \end{cases} \quad \text{Or} \quad \begin{cases} \left(z - (4)23^k\right) = 233^y + (7)23^{2k} \\ \left(z + (4)23^k\right) = 1 \end{cases}$$

From first set of equations we get

$$8(23^k) = 233^y + (7)23^{2k} - 1$$

$$\Longrightarrow 233^y - 1 = 8(23^k) - (7)23^{2k} = 23^k\left(8 - (7)23^k\right)$$

$$\Longrightarrow k=0 \text{ and then } 233^y = 2$$

This is not possible. Hence there is no solution.

Solving the second set of equations we get

$$\Longrightarrow 8(23^k) = 1 - 233^y - (7)23^{2k}$$

$$\Longrightarrow 233^y - 1 = -(7)23^{2k} - 8\left(23^k\right) = 23^k\left(-(7)23^k - 8\right)$$

$$\Longrightarrow 23^k = 1 \text{ and } \left(-(7)23^k - 8\right) = 233^y - 1$$

$$\Longrightarrow k=0 \text{ and } 233^y = -14,$$

This is impossible. Therefore there are no solutions in this case. (5)

Therefore $(x, y, z) = (1, 1, 16)$ is the unique non-negative integer solution of the Diophantine equation $23^x + 233^y = z^2$.

**Corollary 3.1.** Let x, y, w, n>0 be non-negative integers. The Diophantine equation $23^x + 233^y = w^{2n}$ has three solutions $(x, y, w, n) = (1, 1, 2, 4), (1, 1, 4, 2), (1, 1, 16, 1)$.

**Proof:** suppose that x, y and z non-negative integers such that $23^x + 233^y = w^{2n}$. (6)

Let $z = w^n$.Then equation (6) becomes $23^x + 233^y = z^2$ .

Then by theorem 3.1, we have $(x, y, z) = (1, 1, 16)$.

Then we have $w^n = z = 16$, solving this we get

$w = 2, n = 4$ Or $w = 4, n = 2$ Or $w = 16, n = 1$.

Therefore the solutions are $(x, y, w, n) = (1, 1, 2, 4), (1, 1, 4, 2), (1, 1, 16, 1)$.

**Corollary 3.2.** Let x, y, w, n > 0 be non-negative integers. Then the Diophantine equation $23^x + 233^y = w^{4n}$ has two solutions $(x, y, w, n) = (1,1,2,2), (1,1,4,1)$.

**Proof:** suppose that x, y and z non-negative integers such that $23^x + 233^y = w^{4n}$. (7)

Let $z = w^{2n}$.Then equation (7) becomes $23^x + 233^y = z^2$ .

Then by theorem 3.1, we have $(x, y, z) = (1, 1, 16)$.

Then we have $w^{2n} = z = 16$, solving this we get $w = 2, n = 2$ and $w = 4, n = 1$ .

Therefore the solutions are $(x, y, w, n) = (1,1,2,2), (1,1,4,1)$

**Corollary 3.3.** Let x, y, w, n>0 be non-negative integers. Then the Diophantine equation $23^x + 233^y = w^{8n}$ has the unique solution$(x, y, w, n) = (1,1,2,1)$.

**Proof:** suppose that x, y and z non-negative integers such that $23^x + 233^y = w^{8n}$. (8)

Let $z = w^{4n}$.

Then equation (6) becomes $23^x + 233^y = z^2$ .

Then by theorem 3.1, we have $(x, y, z) = (1, 1, 16)$.

Then we have $w^{4n} = z = 16$, solving this we get $w = 2, n = 1$ .

Therefore the solution is $(x, y, w, n) = (1,1,2,1)$

**Theorem 3.2 (Generalization of the theorem 3.1):** Let $x, y, w, k > 0, n > 0$ be non-negative integers and $kn > 1$, $kn$ is an even positive integer. Then

I. The solutions of the Diophantine equation $23^x + 233^y = w^{kn}$ are given by $(x, y, w, n, k) = (1,1,2,4,2), (1,1,4,2,2), (1,1,16,1,2), (1,1,2,2,4), (1,1,4,1,4),$ $(1,1,2,1,8), (1,1,2,8,1), (1,1,4,4,1), (1,1,16,2,1)$.

II. The Diophantine equation $23^x + 233^y = w^{kn}$ has no solutions if $kn \neq 2,4,8$.

**Proof:** Suppose $x, y, w, k > 0, n > 0$ are non-negative integers and $kn > 1$, $kn$ is an even positive integer such that $23^x + 233^y = w^{kn}$. (9)

Let $z = w^{\frac{kn}{2}}$.

Then equation (6) becomes $23^x + 233^y = z^2$ .

Then by theorem 3.1, we have (x, y, z) = (1, 1, 16). Then we have $w^{\frac{kn}{2}} = z = 16$, solving this we get $(w = 2, \mathrm{kn} = 8)$ or $(w = 4, \mathrm{kn} = 4)$ or $(w = 2, \mathrm{kn} = 16)$

$$\text{Hence } (x, y, w) = (1,1,2), (n, k) = (4,2), (2,4), (1,8), (8,1)$$

$$\text{Or } (x, y, w) = (1,1,4), (n, k) = (2,2), (1,4), (4,1)$$

$$\text{Or } (x, y, w) = (1,1,2), (n, k) = (2,1), (1,2).$$

Hence the solutions which agree Corollary 3.1 are

$$(x, y, w, n, k) = (1,1,2,4,2), (1,1,4,2,2), (1,1,16,1,2)$$

The solutions which agree with Corollary 3.2 are

$$(x, y, w, n, k) = (1,1,2,2,4), (1,1,4,1,4)$$

The solution which agrees Corollary 3.3 is

$$(x, y, w, n, k) = (1,1,2,1,8)$$

The other solutions are

$$(x, y, w, n, k) = (1,1,2,8,1), (1,1,4,4,1), (1,1,16,2,1).$$

These are the complete solutions of (9), which exist when $kn = 2,4,8$ only.

It follows that (9) has no solutions when $kn \neq 2,4,8$.

## 4. Open problem

Let p and q be positive prime numbers. We may ask for the set of all solutions $(x, y, z)$ for the Exponential Diophantine Equation $p^x + q^y = z^2$, where $x, y, z$ are non-negative integers.

## 5. Conclusion

In this paper, it is shown that the Exponential Diophantine Equation $23^x + 233^y = z^2$ has exactly one non-negative integer solution $(x, y, z) = (1,1,16)$. Moreover, the Diophantine equation $23^x + 233^y = w^{kn}$ is investigated for non-negative integers in $x, y, w, k > 0, n > 0$ and $kn > 1$, $kn$ is an even positive integer. When $kn \neq 2,4,8$, there are no solutions but for $kn = 2,4,8$, this produces the solutions

$$(x, y, w, n, k) = (1,1,2,4,2), (1,1,4,2,2), (1,1,16,1,2), (1,1,2,2,4), (1,1,4,1,4), (1,1,2,1,8), (1,1,2,8,1), (1,1,4,4,1), (1,1,16,2,1).$$

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**Authors' Contributions.** This is the authors' sole contribution.

## REFERENCES


1. E.Catalan, Note extraite dune lettre adressee a lediteur, *Journal für die Reine und Angewandte Mathematik*, 27(1844), 192.
2. P. Mihailescu, Primary cycolotomic units and a proof of Catalan's conjecture, *J. Reine Angew. Math.*, 2004 (572) (2004) 167-195. https://doi.org/10.1515/crll.2004.048
3. D.Acu, On a Diophantine equation $2^x + 5^y = z^2$, *General Mathematics*, 15(4) (2007) 145-148.
4. J.F.T.Rabago, On two Diophantine equations $3^x + 19^y = z^2$ and $3^x + 91^y = z^2$, *Int. J. of Math. and Scientific Computing*, 3 (2013) 28-29.
5. S.Chotchaisthit, On the Diophantine equation $2^x + 11^y = z^2$, *Maejo Int. J. Sci. Technol.*, 7(2) (2013) 291-293.
6. N.Burshtein, On the Diophantine equation $p^x + q^y = z^2$, *Annals of Pure and Applied Mathematics*, 13(2) (2017) 229-233.
7. C.G.Rao, On the Diophantine equation $3^x + 7^y = z^2$, *EPRA International Journal of Research and Development*, 3(6) (2018) 93–95.
8. N.Burshtein, On solutions of the Diophantine equation $11^x + 23^y = z^2$ with consecutive positive integers x, y, *Annals of Pure and Applied Mathematics*, 20(2) (2019) 57-61.
9. S.Asthana and M.M.Singh, On the Diophantine equation $53^x + 143^y = z^2$, *Annals of Pure and Applied Mathematics*, 19(2) (2019) 193-196.
10. K.Srimud and S.Tadee, On the Diophantine equation $3^x + b^y = z^2$, *International Journal of Mathematics and Computer Science*, 18(1) (2023) 137-142.
11. N.Burshtein, Solutions of the Diophantine equation $p^x + (p+6)^y = z^2$ when p, (p+ 6) are primes and x+ y= 2, 3, 4, *Annals of Pure and Applied Mathematics*, *17*(1) (2018) 101-106.
12. N.Burshtein, On the class of the Diophantine equations $(10K + A)^x + (10M + A)^y = z^2$ when A= 1, 3, 7, 9 with positive integers x, y, z, *Annals of Pure and Applied Mathematics*, 21(2) (2020) 103-111.
13. W.Orosram, On the Diophantine equation $(p + n)^x + p^y = z^2$ where p and p+ n are prime numbers, *European Journal of Pure and Applied Mathematics*, *16*(4) (2023) 2208-2212.
14. N.Burshtein, On the Diophantine equation $2^x + p^y = z^2$, *Annals of Pure and Applied Mathematics*, 16(2) (2018) 471-477.