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The Co-Intersection Graphs of Ideals of Rings

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Abstract. Let $I^{\star}(R)$ be the set of all nontrivial left ideals of ring R. The Co-intersection graph of ideals of R, denoted by $\Omega(R)$, is a simple undirected graph with the vertex set $I^{\star}(R)$, and two distinct vertices I and J are adjacent if and only if $I+J\neq R$. This paper derives a sufficient and necessary condition for $\Omega(R)$ to be a complete graph. Among other results, we determine the domination number of $\Omega(\mathbb{Z}_n)$. Further, the good and excellent decision numbers of $\Omega(\mathbb{Z}_n)$ are studied in the paper.

Keywords: Co-intersection graph, Domination number, Decision number.

AMS Mathematics Subject Classification (2010): 05C25, 05C40, 05C45, 05C69

1. Introduction

The concept of associating a graph to a ring was initially proposed in [5]. He let all ring elements be vertices of the graph and was interested mainly in coloring. In [4], the zero-divisor graph, whose vertices are nonzero zero-divisors, was introduced and investigated by Anderson and Livingston. Many papers have been written about how to assign a graph to a ring; for instance, see [1, 2, 3, 4, 11, 12]. Also, several authors have investigated the intersection and co-intersection graphs of algebraic structures such as groups, rings, and modules, see [2, 7, 9, 10]. The co-intersection graph of submodules is introduced in [9]. Further, some results on the Co-Intersection graphs of ideals of rings are presented in [14]. This is how the paper is structured: Section 2 introduces some definitions and preliminaries. We devote Section 3 to studying for completeness of the co-intersection graph. Also, we present some results about the domination number of co-intersection graph $\Omega(R)$ in this section. Finally, the good decision number and the excellent decision number of $\Omega(\mathbb{Z}_n)$ are studied in Section 4.

2. Preliminaries

The definitions of ring theory and graph theory are provided in this section. In addition, we introduce the Co-intersection graph of a ring and discuss some fundamental concepts related to rings and maximal left ideals.

In this paper, let R denote a ring. We mean from a nontrivial ideal of R is a nonzero proper left ideal of R. By $I^{\star}(R)$, we denote the set of all nontrivial left ideals of R. A ring R is said to be *local* if it has a unique maximal left ideal. The ring of $n \times n$ matrices over R is denoted by $M_n(R)$. The sets of all nonzero maximal left ideals of R and all nonzero minimal left ideals of R are denoted by $Max(R)$ and $Min(R)$, respectively.

A graph G is an ordered pair $G = (V, E)$, that consists of a nonempty set V of vertices, and a set $E \subseteq [V]^2$ of edges, where $[V]^2$ is the set of all 2-element subsets of V. Two vertices $u, v \in V$ are *adjacent* if $uv \in E$ (for simplicity we use uv instead of subset $\{u, v\}$). The *neighbourhood* of a vertex $u \in V$ is $N(u) = \{v \in V | uv \in E\}$, and the *closed neighbourhood* of u is $N[u] = N(u) \cup \{u\}$. The degree of a vertex u in a graph G is the size of set $N(u)$, which is denoted by $deg(u)$. We denote by $\Delta(G)$ the maximum degree of the vertices of G. A complete graph of order n, denoted by K_n, is a graph in which any two distinct vertices are adjacent. A *null graph* is a graph containing no edges. In the graph theory, a *dominating set* for a graph $G = (V, E)$ is a subset D of V such that every vertex not in D is adjacent to at least one member of D. The *domination number*$\gamma(G)$ is the number of vertices in the smallest dominating set for G. If $G = (V, E)$ is a finite graph, define $f(U) = \sum_{u \in U} f(u)$, for a function $f: V \to \{-1,1\}$ and $U \subseteq V$. A function $f: V \to \{-1,1\}$ is called a *good function* of G, if $f\big(N(v)\big) \geq 1$, for each $v \in V$. The *good decision number* of G, which is denoted by $\lambda(G)$, is the minimum value of $f(V)$, taken over all good function f. The function f is called an *excellent function*, if $f(N[v]) \geq 1$ for each $v \in V$. The minimum value of $f(V)$, taken over all excellent function f, is called the *excellent decision number* of G, and denoted by $\overline{\lambda(G)}$.

Definition 2.1. *The Co-intersection graph $\Omega(R)$ of ring R, is an undirected simple graph whose the vertex set $V\big(\Omega(R)\big) = I^{\star}(R)$ is a set of all nontrivial ideals of R and two distinct vertices I, J are adjacent if and only if $I + J \neq R$.*

Remark 2.2. *Let $R = \mathbb{Z}_n$ be the integers modulo n. Suppose that m_1 and m_2 are two factors of n. So $< m_1 > + < m_2 > = < (m_1, m_2) >$, where (m_1, m_2) is the greatest common divisor of m_1, m_2.*

Example 3.3. *Suppose that $R = Z_{225}$. Then $I^{\star}(R) = \{< 3 >, < 5 >, < 9 >, < 15 >, < 25 >, < 45 >, < 75 >\}$ and the co-intersection graph $\Omega(R)$ is as follow:*

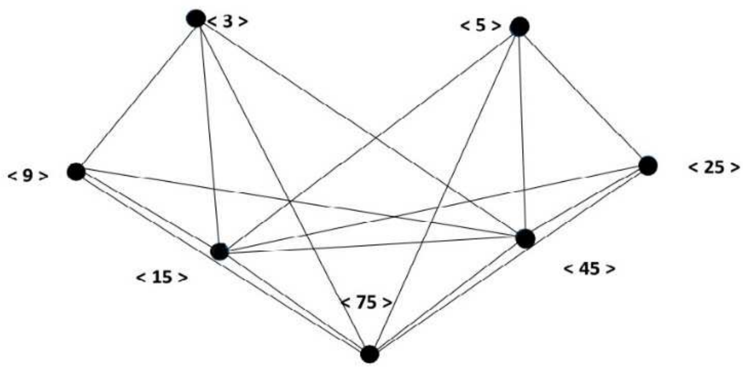

Figure 1: The Co-intersection Graph $\Omega(Z_{225})$.

3. The Domination Number and Completeness

In this section, we characterize the domination number of co-intersection graph $\Omega(\mathbb{Z}_n)$, and we present some results for the domination number of $\Omega(R)$; also, we study the total dominating set of $\Omega(\mathbb{Z}_n)$. Further, we derive a sufficient and necessary condition for $\Omega(R)$ to be a complete graph. Furthermore, we determine the values of n for which $\Omega(\mathbb{Z}_n)$ is a complete graph.

Proposition 3.1. *Let $n = p_1^{\alpha_1} p_2^{\alpha_2} \cdots p_k^{\alpha_k}$, where p_i's are all distinct prime numbers, and also $G = \Omega(\mathbb{Z}_n)$. Then the domination number $\gamma(G)$ is two, if $\alpha_i = 1$ for all $1 \leq i \leq k$; and otherwise $\gamma(G) = 1$.*

Proof: At first, suppose that $\alpha_{i_1} > 1$, for some $1 \leq i_1 \leq k$. We show that the set $\{I =< p_1 p_2 \cdots p_k >\}$ is a dominating set for G. As $\alpha_{i_1} > 1$, then $p_1 p_2 \cdots p_k \neq n$ and therefore I is an nontrivial ideal of $\mathbb{Z}_n$. Now assume that $J =< m >$ is an nontrivial ideal of $\mathbb{Z}_n$ different from I, where m is a factor of n. It is obvious that the greatest common divisor of m and $p_1 p_2 \cdots p_k$ is grater than one. Then $I + J =< (m, p_1 p_2 \cdots p_k) > \neq \mathbb{Z}_n$. Hence I and J are adjacent and $\gamma(G) = 1$.

Now suppose that $\alpha_i = 1$ for all $1 \leq i \leq k$. Let $a_1 = p_1 p_2 \cdots p_{k-1}, a_2 = p_2 p_3 \cdots p_k$, then $I_1 =< a_1 >$ and $I_2 =< a_2 >$ are two nontrivial ideals of $\mathbb{Z}_n$. Assume that $J =< m >$ is an nontrivial ideal of $\mathbb{Z}_n$ different from I_1, I_2, where m is a factor of n. At least one of the greatest common divisor, (m, a_1) or (m, a_2) is grater than one. Therefore there is an edge between J and one of the vertices I_1, I_2. Hence, $\{a_1, a_2\}$ is a dominating set for G and $\gamma(G) \leq 2$. On the other hand, because $\alpha_i = 1$ for all $1 \leq i \leq k$, for each nontrivial ideal $< m >$ of $\mathbb{Z}_n$, there is nontrivial ideal $< \frac{n}{m} >$, such that $< m > + < \frac{n}{m} >< 1 \geq \mathbb{Z}_n$. Then $\gamma(G) > 1$. Then $\gamma(G) = 2$.

Proposition 3.2. *Let $R = R_1 \times \cdots \times R_n$ and $G_i = \Omega(R_i)$. Then $\gamma(\Omega(R)) = \infty$ if $\gamma(G_i) = \infty$ for each $1 \leq i \leq n$, otherwise $\gamma(\Omega(R)) = min\{\gamma(G_i) | 1 \leq i \leq n\}$.*

Proof: If $\gamma(G_i) = \infty$ for each $1 \leq i \leq n$ then $\gamma(\Omega(R)) = \infty$. Suppose that $\gamma_0 = \gamma(G_{i_0}) = min\{\gamma(G_i) | 1 \leq i \leq n\}$ and $D_{i_0} = \{I_1, \cdots, I_{\gamma_0}\}$ is a dominating set for G_{i_0}. Thus $D = \{0 \times \cdots \times I_j \times \cdots \times 0 | I_j \in D_{i_0}, 1 \leq j \leq \gamma_0\}$ is a dominating set for G and thus $\gamma(\Omega(R)) \leq \gamma_0$. On the other hand, as $R_1 \times \cdots \times I \cdots \times R_n$ is a left ideal of R, for each left ideal I of R_{i_0}, thus $\gamma(\Omega(R)) \geq \gamma_0$. Therefore $\gamma(\Omega(R)) = \gamma_0$.

Lemma 3.3. *Let R be a ring with unity element 1 and $G = \Omega(R)$. Then $\gamma(G) \leq |Max(R)|$ and the equality is hold if $Max(R) \cap Min(R) \neq \emptyset$.*

Proof: $Max(R)$ is a dominating set for G, as if I is a left ideal of R, then either $I \in Max(R)$ or there is a maximal left ideal m contain I and thus $I + m \neq R$. Also, if $Max(R) \cap Min(R) \neq \emptyset$, then ,$G$ is a null graph and thus $\gamma(G) = |Max(R)|$.

Example 3.4. *Let Z be the ring of integers. $Max\ (\mathbb{Z}) = \{< p > | for\ prime\ number\ p\}$ is a dominating set for $\Omega(\mathbb{Z})$. As, the number of prime numbers is infinite and $< m > + < p >= \mathbb{Z}$ for each prime number $p \nmid m, m \in Z$, thus $\gamma(\mathbb{Z}) = |Max(R)| = \infty$. This example shows that the converse of Lemma 3.3 is not true.*

A dominating set D in G is a *total dominating set* if $G[D]$ has no isolated vertex. It is obvious that if D is a total dominating set, then it is a dominating set and also $|D| \geq 2$. In the next proposition, we show that $\Omega(Z_n)$ has a total dominating set of size 2 for each $n = p_1^{\alpha_1} p_2^{\alpha_2} \cdots p_k^{\alpha_k}$ where $\sum_{i=1}^{k} \alpha_i \geq 3$.

Proposition 3.5. *Let $n = p_1^{\alpha_1} p_2^{\alpha_2} \cdots p_k^{\alpha_k}$, where p_i's are all distinct prime numbers and $\sum_{i=1}^{k} \alpha_i \geq 3$. Then $\Omega(\mathbb{Z}_n)$ has a total dominating set of size 2.*

Proof: Let $a_1 = p_1 p_2 \cdots p_{k-1}, a_2 = p_{k-1} p_k$ for $k \geq 3$, $a_1 = p_1 p_2, a_2 = p_2$ for $k = 2$ and $a_1 = p_1, a_2 = p_1^2$ for $k = 1$. Then $D = \{I_1 =< a_1 >, I_2 =< a_2 >\}$ is a total dominating set for $\Omega(\mathbb{Z}_n)$.

In the following, we provide a necessary and sufficient conditions for complete graph $\Omega(R)$.

Proposition 3.6. *Let R be a ring with unity element 1. Co-intersection graph $\Omega(R)$ is complete if and only if R has a unique maximal left ideal. In other words, co-intersection graph $\Omega(R)$ is complete if and only if R is a local ring ($|Max(R)| = 1$).*

Proof: Suppose that m is a unique maximal left ideal of R.

Now assume that J_1, J_2 are two arbitrary different proper left ideals of R. Then $J_1 \subset m$ and $J_2 \subset m$; therefore $J_1 + J_2 \subset m \neq R$. Hence J_1, J_2 are adjacent in $\Omega(R)$ and $\Omega(R)$ is a complete graph.

Conversely, let $\Omega(R)$ be a complete graph. Suppose that m is a maximal left ideal of R. The ideal m is a unique maximal left ideal of R. Otherwise, there are at least two maximal left ideals, and according to [14, Lemma 3.1], there are two non-adjacent vertices in $\Omega(R)$, and then $\Omega(R)$ is not a complete graph. Hence m is unique.

Example 3.7. *Ring $\mathbb{Z}$ has more than one maximal ideal. Then $\Omega(\mathbb{Z})$ is not complete.*

Example 3.8. *Suppose that $\mathbb{F}$ is a field, Then:*

- *Let $R = \mathbb{F}[X]$ be the polynomial ring over field $\mathbb{F}$. Then $\Omega(R)$ is not complete.*
- *Let $R = M_n(\mathbb{F})$ be the ring of $n \times n$ matrices over field $\mathbb{F}$. Then $\Omega(R)$ is not complete.*

According to the Hilbert basis theorem, ring $R = \mathbb{F}[X]$ is a Noetherian ring, and $< x >, < x + 1 >$ are two maximal ideal of R. Then $\Omega(R)$ is not complete.

As $\mathbb{F}$ is a field, then $R = \mathbb{M}_n(\mathbb{F})$ is a left Noetherian ring, and

$m_1 = \{[a_{ij}]_{n\times n} | 1 \leq i, j \leq n, a_{ij} \in \mathbb{F}, a_{i1} = 0\}$,

$m_2 = \{[b_{ij}]_{n\times n} | 1 \leq i, j \leq n, b_{ij} \in \mathbb{F}, b_{i2} = 0\}$

are two maximal left ideal of R. Then $\Omega(R)$ is not complete.

4. The decision number of $\Omega(\mathbb{Z}_n)$

The bad decision number and the nice decision number of $\Omega(\mathbb{Z}_n)$ have been investigated. In this section, the good decision number and the excellent decision number of $G = \Omega(\mathbb{Z}_n)$ are investigated for each n.

At first, some lemma's are presented in the following, and finally, the results are combined to a single theorem.

Lemma 4.1. *Let $n = p^{\alpha}$, $\alpha \geq 3$, and also $G = \Omega(\mathbb{Z}_n)$. Thus,*

$$\lambda(G) = \begin{cases} 2, & \text{for odd } \alpha, \\ 3, & \text{for even } \alpha, \end{cases} \text{ and } \overline{\lambda(G)} = \begin{cases} 2, & \text{for odd } \alpha, \\ 1, & \text{for even } \alpha. \end{cases}$$

Proof: The proof is similar to $[14, Lemma\ 4.1]$

Lemma 4.2. *Let $k \geq 3$, $n = p_1 p_2 \cdots p_k$, where p_i's are all distinct prime numbers, and $G = \Omega(\mathbb{Z}_n)$. We have $\lambda(G) \in \{0,2,4\}, \overline{\lambda(G)} \in \{0,2\}$.*

Proof: Define the function $f: V \to \{-1,1\}$ as:

$$f(< a >) = \begin{cases} 1, & if\ p_1 | a, \\ 1, & a = p_2 \ldots p_k\ or\ a = p_2 \\ -1 & otherwise \end{cases}$$

Assume that $< a >$ is a nontrivial ideal of $\mathbb{Z}_n$, and $a = p_1^{\alpha_1} p_2^{\alpha_2} \cdots p_k^{\alpha_k}$. Let $A = \{i | a_i \neq 0\}$.

- If $p_1 | a$, then there are at least $2^{k-1} - 2$ elements of $N(< a >)$ with value 1 and at most $2^{k-1} - 3$ elements of $N(< a >)$ with value -1 under the function f. Therefore, $f(N(< a >)) \geq 1$ and $f(N[< a >]) \geq 2$ as $f(< a >) = 1$.
- If $p_1 \nmid a$, then there are at least $2^{k-1} - 2^{k-|A|-1}$ elements of $N(< a >)$ with value 1 and at most $2^{k-1} - 2^{k-|A|-1} - 2$ elements of $N(< a >)$ with value -1 under the function f. So, $f(N(< a >)) \geq 2$ and $f(N[< a >]) \geq 1$.

Hence, f is a good (and excellent) function and $f(V) = 4$, thus $\lambda(G), \overline{\lambda(G)} \leq 4$.
Similarly, it can be proved that the function

$$g(< a >) = \begin{cases} 1, & if\ p_1 | a\,, \\ 1, & if\ a = p_2 \ldots p_k\,, \\ -1 & otherwise. \end{cases}$$

is an excellent function and hence $\overline{\lambda(G)} \leq 2$.
Now let $v_0 = p_2 \cdots p_k$. we have $N(< v_0 >) = V(G) \backslash \{< v_0 >, < p_1 >\}$. If f is a good function, then $f(N < v_0 >) \geq 2$, because of $|N(< v_0 >)|$ is even. Also, $f(N[< v_0 >])$ is at most equal to 1 for an excellent function f. Thus, $f\big(V(G)\big) \geq 0$ for any good or excellent function f. Further, $\lambda(G), \overline{\lambda(G)}$ are both even, as $|V(G)|$ is even. Hence, $\lambda(G) \in \{0,2,4\}, \overline{\lambda(G)} \in \{0,2\}$.

In the next two lemma's we show that $\lambda\big(\Omega(Z_n)\big) = 3, \overline{\lambda\big(\Omega(Z_n)\big)} = 1$, when $k \geq 2$, $n = p_1^{\alpha_1} p_2^{\alpha_2} \cdots p_k^{\alpha_k}$, and α_i's are all even numbers.

Lemma 4.3. *Let $k \geq 2$, $n = p_1^{\alpha_1} p_2^{\alpha_2} \cdots p_k^{\alpha_k}$, where p_i's are all distinct prime numbers, α_i's are all even numbers, and $G = \Omega(\mathbb{Z}_n)$. Then, $\lambda(G) \leq 3$, and $\overline{\lambda(G)} \leq 1$.*

Proof: Let $m_i = \frac{\alpha_i}{2}$ for each $1 \leq i \leq k$. Define the function $f: V \to \{-1,1\}$ as:

$$f(< a >) = \begin{cases} 1, & if\ p_1^{a_1} \ldots p_i^{a_i} | a\ and\ p_{i+1}^{m_{i+1}} | a\ and\ p_{i+1}^{a_{i+1}} a \nmid a\ for\ some\ 0 \leq i \leq k-1 \\ 1, & if\ a_1 = 2\ and\ a = p_1^2, \\ 1, & if\ a_1 \neq 2\ and\ a = p_1 p_2 \ldots p_k, \\ -1 & otherwise \end{cases}$$

Suppose that $a >$ is a nontrivial ideal of $\mathbb{Z}_n$, and $a = p_1^{a_1} p_2^{a_2} \cdots p_k^{a_k}$. Let $A = \{i | a_i \neq 0\}$, and $t = min\{i \vee a_i \neq 0\}$.

Case 1. If $p_1 \nmid a$, then $f(< a >) = -1$, further, if $\alpha_1 = 2$, then

$$X = \sum_{i=t}^{k} m_i \prod_{j=i+1}^{k} (\alpha_j + 1) + \sum_{i=1}^{t-1} m_i \left(\prod_{j\in A, j>i} (\alpha_j + 1) - 1 \right) \prod_{j\notin A, j>i} (\alpha_j + 1)$$

elements of $N[< a >]$ have value 1 under the function f, and

$$Y = \sum_{i=t}^{k} m_i \prod_{j=i+1}^{k} (\alpha_j + 1) - \prod_{j\notin A, j>t} (\alpha_j + 1) + \sum_{i=1}^{t-1} m_i \left(\prod_{j\in A, j>i} (\alpha_j + 1) - 1 \right) \prod_{j\notin A, j>i} (\alpha_j + 1)$$

elements of $N[< a >]$ have value -1 under the function f. Because of, $f(< a >) = -1$, thus X elements of $N(< a >)$ have value 1, and $Y - 1$ elements of $N(< a >)$ have value -1. If $\alpha_1 \neq 2$, then $X + 1$ elements of $N(< a >)$ have value 1, and $Y - 2$ elements of $N(< a >)$ have value -1. Thus,

$$f\big(N(< a >)\big) = \begin{cases} X - Y + 1 = \sum_{i=t}^{k} \prod_{j\notin A, j>t} (a_j + 1) + 1, & if\ a_1 = 2, \\ X - Y + 3 = \sum_{i=t}^{k} \prod_{j\notin A, j>t} (a_j + 1) + 3, & if\ a_1 \neq 2. \end{cases}$$

Consequently, $f(N(< a >)) \geq 2$ and $f(N[< a >]) \geq 1$. Hence, f is both a good function and a excellent function.

Case 2. *If* $p_1 | a$, *then* $t = 1$. *If* $f(< a >) = 1$, *then* $f\big(N(< a >)\big) = X - (Y - 1) = \sum_{i=1}^{k} \prod_{j\notin A, j>t} (a_j + 1) + 1 \geq 2$, *and* $f(N[< a >]) \geq 3$. *If* $f(< a >) = -1$, then $f\big(N(< a >)\big) = X + 1 - (Y - 2) = \sum_{i=1}^{k} \prod_{j\notin A, j>t} (a_j + 1) + 3 \geq 4$, and $f(N[< a >]) \geq 3$. By the definition of the function f

$$Z + 1 = \sum_{i=1}^{k} m_i \prod_{j=i+1}^{k} (\alpha_j + 1) + 1$$

vertices of G have value 1, and $Z - 2$ elements of $N(< a >)$ have value -1. Hence, $\lambda(G), \overline{\lambda(G)} \leq 3$.

Similarly, it can be concluded that the function

$$g(< a >) = \begin{cases} 1, & if\ p_1^{a_1} \dots p_i^{a_i} | a\ and\ p_{i+1}^{m_{i+1}} | a\ and\ p_{i+1}^{a_{i+1}} \nmid a\ for\ some\ 0 \leq i \leq k-1, \\ -1, & otherwise \end{cases}$$

is an excellent function over $V(G)$ and hence, $\overline{\lambda(G)} \leq 1$

Lemma 4.4. *Let* $k \geq 2$, $n = p_1^{\alpha_1} p_2^{\alpha_2} \cdots p_k^{\alpha_k}$, α_i*'s are all even numbers, and* $G = \Omega(\mathbb{Z}_n)$. *Then,* $\lambda(G) = 3$, *and* $\overline{\lambda(G)} = 1$.

Proof: Assume that f is a good function over $V(G)$. If there is $< v > \in G$ of maximum degree $\Delta(G) = n - 1$, such that $f(< v >) = 1$, then $f\big(V(G)\big) \geq 3$, because of $|N(< v >)| = |V(G)\{< v >\}|$ is an even number. For $k = 2$ more than of half of all vertices are of maximum degree $\Delta(G) = n - 1$ thus, there exist a $< v > \in G$ of maximum degree $\Delta(G) = n - 1$, such that $f(< v >) = 1$. Hence, $\lambda(G) \geq 3$ for $k = 2$. So suppose that $k \geq 3$ and $f(< u >) = -1$ for each $< v > \in G$ of maximum degree $\Delta(G) = n - 1$. Suppose that $\alpha_1 \leq \alpha_2 \leq \cdots \leq \alpha_k$, and let $a_1 = p_1, a_2 = p_2, a_3 = p_3 \cdots p_k$. There are at least $\frac{\alpha_1 \prod_{i=2}^{k}(\alpha_i+1)}{2}$ elements of $N(< a_1 >)$ with value 1 under the function f. As, $f(< u >) = -1$ for each $< v > \in G$ of maximum degree $\Delta(G) = n - 1$, thus the number of vertices in $\{N(< a_1 >) \cup N(< a_2 >) \cup N(< a_3 >)\}$ with value 1 is at least

$$X = \frac{\alpha_1 \prod_{i=2}^{k}(\alpha_i + 1)}{2} + \frac{\alpha_2 \prod_{\substack{i=1 \\ i\neq 2}}^{k}(\alpha_i + 1)}{2} + \frac{\left(\prod_{i=1}^{k}(\alpha_i + 1) - 1\right)(\alpha_1 + 1)(\alpha_2 + 1)}{2}$$
$$-\alpha_1\alpha_2 \prod_{i=3}^{k}(\alpha_i + 1) - (\alpha_2 + 1)\prod_{\substack{i=1 \\ i\neq 2}}^{k} \alpha_i - (\alpha_1 + 1)\prod_{i=2}^{k} \alpha_i + 3\prod_{i=1}^{k} \alpha_i\,.$$

With some manipulation we get $X \geq \frac{\prod_{i=1}^{k}(\alpha_i+1)+1}{2} = \frac{|V(G)|+3}{2}$, and hence $f\big(V(G)\big) \geq 3$. Consequently, $\lambda(G) \geq 3$.

Now suppose that f is an excellent function and $a = p_1 \cdots p_k$. We have $N[< a >] = V(G)$ and thus $|N[< a >]|$ is an odd number and then, $f(N[< a >]) \geq 1$, as f is an excellent function.

On the other side $\lambda(G) \leq 3$ and $\overline{\lambda(G)} \leq 1$ from Lemma 4.3. Therefore $\lambda(G) = 3$ and $\overline{\lambda(G)} = 1$

Lemma 4.5. *Let $k \geq 2$, α_1 be an odd number, $n = p_1^{\alpha_1} p_2^{\alpha_2} \cdots p_k^{\alpha_k}$, and also $G = \Omega(Z_n)$. If there exist an $1 \leq s \leq k$ such that $\alpha_s > 1$ then, $\lambda(G) \leq 2$ and $\overline{\lambda(G)} \leq 2$.*

Proof: Let $m = \frac{\alpha_1+1}{2}$. Define the function $f: V \to \{-1,1\}$ as:

$$f(< a >) = \begin{cases} 1, & if\ a_1 = 1\ and\ ((p_1|a\ \ and\ \ a_1 \neq p_1)\ or\ a = p_s\ \ or\ a = \dfrac{n}{p_1}) \\ 1, & if\ a_1 \neq 1\ and\ (p_1^m|a\ \ or\ \ a_1 = p_1 \dots p_k), \\ -1, & otherwise \end{cases}$$

Suppose that $< a >$ is a nontrivial ideal of $\mathbb{Z}_n$, and $a = p_1^{a_1} p_2^{a_2} \cdots p_k^{a_k}$. Let $A = \{i | a_i \neq 0\}$.

Case 1. $\alpha_1 \neq 1$:

- If $a_1 = 0$, thus $f(< a >) = -1$. There are
$$X = m\left(\prod_{i=2}^{k}(\alpha_i + 1) - \prod_{i\notin A, i\neq 1}(\alpha_i + 1)\right)$$
elements in $N(< a >)$ with value 1 under f. Also, There are $X - 2$ elements in $N(< a >)$, with value -1 under f. Hence $f\big(N(< a >)\big) = 2$, and $f(N[a]) = 1$.
- If $a_1 \neq 0$:
If $f(< a >) = -1$, then there are $Z = m\prod_{i=2}^{k}(\alpha_i + 1)$ elements in $N(< a >)$ with value 1 and

$$W = (m-1)\prod_{i=2}^{k}(\alpha_i+1) - 2 + \prod_{i=2}^{k} - \prod_{i\notin A}(\alpha_i+1)$$

elements in $N(< a >)$, with value -1 under f. Hence, $f\big(N(< a >)\big) = 2 + \prod_{i\notin A}(\alpha_i+1) \geq 3$, and $f(N[a]) \geq 2$.
If $f(< a >) = 1$, then there are $Z-1$ elements in $N(< a >)$ with value 1 and $W+1$ elements in $N(< a >)$, with value -1 under f. Hence, $f\big(N(< a >)\big) = \prod_{i\notin A}(\alpha_i+1) \geq 1$, and $f(N[a]) \geq 2$.

Case 2. $\alpha_1 = 1$:

- If $a_1 = a_s = 0$. There are $X = \prod_{i=2}^{k}(\alpha_i+1) - \prod_{i\notin A, i\neq 1}(\alpha_i+1)$ elements in $N(< a >)$ with value 1 and $X-2$ elements in $N(< a >)$ with value -1 under f. Hence, $f\big(N(< a >)\big) = 2$, and $f(N[< a >]) = 1$.
- If $a_1 \neq 0, a_s = 0$. At first suppose that $f(< a >) = 1$ thus $a \neq p_1$. There are $\prod_{i=2}^{k}(\alpha_i+1) - 2$ elements in $N(< a >)$ with value 1 and $\prod_{i=2}^{k}(\alpha_i+1) - 2$ elements in $N(< a >)$ with value 1 and $\prod_{i=2}^{k}(a_i+1) - \prod_{i\notin A}(a_i+1)$ elements in $N(< a >)$ with value -1 under f. Therefore, $f\big(N(< a >)\big) = \prod_{i\notin A}(a_i+1) - 2 \geq 1$ and $f(N[< a >]) \geq 2$ as $a_s \geq 2$.
Now suppose that $f(< a >) = -1$ thus $a = p_1$. Similarly, $f\big(N(< a >)\big) = \prod_{i\neq 1}(a_i+1) - 2 \geq 1$. Furthermore, if $k \geq 3$ then, $f\big(N(< a >)\big) = \prod_{i\neq 1}(a_i+1) - 2 \geq 3 * 2 - 2 = 4$ and $f(N[< a >]) \geq 3$, and if $k = 2, \alpha_s \geq 3$ then, $f\big(N(< a >)\big) \geq 4 - 2 = 2, f(N[< a >]) \geq 1$.
- If $a_1, a_s \neq 0$. Thus, $f(< a >) = 1$ and there are $\prod_{i=2}^{k}(\alpha_i+1) - 1$ elements in $N(< a >)$ with value 1 and $\prod_{i=2}^{k}(\alpha_i+1) - \prod_{i\notin A}(\alpha_i+1) - 1$ elements in $N(< a >)$ with value -1 under f. Hence, $f\big(N(< a >)\big) = \prod_{i\notin A}(a_i+1) \geq 1$, and $f(N[< a >]) \geq 2$.
- If $a_1 = 0, a_s \neq 0$. Similar to the previous items discussed so far, if $f(< a >) = 1$, then $f(N(< a >)) \geq 2$, and $f(N[< a >]) \geq 3$, and if $f(< a >) = -1$, then $f(N(< a >)) \geq 4)$, and $f(N[< a >]) \geq 3$.

Therefore, f is a good function over G, also, f is an excellent function if $k \neq 2$ or $\alpha_s \geq 3$. Further, it is easy to check that $\overline{\lambda(\Omega(\mathbb{Z}_{p_1 p_2^2}))} = 2$. Hence, $\lambda(G) \leq 2$ and $\overline{\lambda(G)} \leq 2$ as, $f(V) = 2$.

Lemma 4.6. *Let $k \geq 2$, α_1 be an odd number, $n = p_1^{\alpha_1} p_2^{\alpha_2} \cdots p_k^{\alpha_k}$, and also $G = \Omega(\mathbb{Z}_n)$. If there exist an $1 \leq s \leq k$ such that $\alpha_s > 1$ then, $\lambda(G) = \overline{\lambda(G)} = 2$.*

Proof: Suppose that f is an excellent function and $a = p_1 \cdots p_k$. We have $N[< a >] = V(G)$ and thus $|N[< a >]|$ is an even number and then, $f(N[< a >]) \geq 2$, as f is an excellent function. Hence, $\overline{\lambda(G)} = 2$ according to the Lemma 4.5.
Further, similar to the proof of Lemma 4.4, it can be shown that $\lambda(G) = 2$.
Finally, the following theorem can immediately be concluded from the above discussions.

Theorem 4.7. *Let $n = p_1^{\alpha_1} p_2^{\alpha_2} \cdots p_k^{\alpha_k}$, where p_i's are all distinct prime numbers, and also $G = \Omega(\mathbb{Z}_n)$. If $n \neq p, p^2, pq$ then,*

$$\lambda(G)=\begin{cases}0 \text{ or } 2 \text{ or } 4 & if\ a_i=1\ for\ 1\leq i\leq k\\ 3 & if\ a_i\ is\ an\ even\ number\ for\ all\ 1\leq i\leq k\\ 2 & otherwise\end{cases}$$

Further, if $n\neq p$ *then,*

$$\overline{\lambda(G)}=\begin{cases}0 \text{ or } 2 & if\ a_i=1\ for\ 1\leq i\leq k\\ 1 & if\ a_i\ is\ an\ even\ number\ for\ all\ 1\leq i\leq k\\ 2 & otherwise\end{cases}$$

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