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New Subclass of Analytic Functions Associated with Multiplier Transformation

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Abstract: In this work, we introduce and study a new subclass of analytic functions defined by a multiplier transformation and obtained coefficient estimates, growth and distortion theorems, radii of starlikeness, convexity and close-to-convexity are obtained. Furthermore, we obtained integral means inequalities for the class.

Keywords: Analytic, coefficient bounds, starlike, distortion**,** integral means inequality**.**

AMS Mathematics Subject Classification (2010): 30C45

1. Introduction

Complex analysis is one of the major disciplines nowadays due to its numerous applications not just in mathematical science, but also in other fields of study. Among the other disciplines, geometric function theory is an intriguing area of complex analysis that involves the geometrical characteristics of analytical functions. It has been observed that this area is crucial to applied mathematics, particularly in fields like engineering, electronics, nonlinear integrable system theory, fluid dynamics, modern mathematical physics, partial differential equation theory, etc. The foundation of function theory is the theory of univalent functions, and as a consequence of its wide application, new fields of research have emerged with a variety of fascinating results. Below, in the first section, we briefly discuss the basics of function theory, which will help in understanding the terminology used in our results.

Let A denote the class of functions f of the form

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n \qquad (1)$$

which are analytic in the open unit disk $E = \{z \in \mathbb{C} : |z| < 1\}$

.

A function f in the class A is said to be in the class $ST(\alpha)$ of starlike functions of order α in *E*, if it satisfy the inequality

$$\operatorname{Re}\left\{\frac{zf'(z)}{f(z)}\right\}>\alpha, \qquad (0\le\alpha<1), \quad (z\in E) \tag{2}$$

Note that $ST(0)=ST$ is the class of starlike functions.

Denote by T the subclass of A consisting of functions f of the form

$$f(z)=z-\sum_{n=2}^{\infty}a_n z^n \quad (a_n\ge 0). \tag{3}$$

This subclass was introduced and extensively studied by Silverman [15].

A function f(z) is uniformly convex (uniformly starlike) in U if $f(z)$ is in CV (ST) and has the property that for every circular arc γ contained in U, with center ξ also in U, the arc $f(\gamma)$ is convex (starlike) with respect to $f(\xi)$. The class of uniformly convex functions denoted by UCV and the class of uniformly starlike functions by UST (for details see [5,6]). It is well known from [10, 13] that

$$f\in UCV \Leftrightarrow \left|\frac{zf''(z)}{f'(z)}\right|\le \Re e\left\{1+\frac{zf''(z)}{f'(z)}\right\}$$

In [13], Ronning introduced a new class of starlike functions related to UCV and defined as

$$f\in S_p \Leftrightarrow \left|\frac{zf'(z)}{f(z)}-1\right|\le \Re e\left\{\frac{zf'(z)}{f(z)}\right\}$$

Note that $f(z) \in UCV \Leftrightarrow zf'(z)\in S_p$.

Further Ronning generalized the class S_p by introducing a parameter α, $-1\le\alpha<1$,

$$f\in S_p(\alpha) \Leftrightarrow \left|\frac{zf'(z)}{f(z)}-1\right|\le \Re e\left\{\frac{zf'(z)}{f(z)}-\alpha\right\}$$

Cho and Srivastava [4] introduced the linear operator $\Im_\lambda^m f(z)$: A→ A by

$$\Im_\lambda^m f(z)=z+\sum_{n=2}^{\infty}\phi_n(\lambda,m)a_n z^n, -1<\lambda\le 1, and\ m\in N_0=\{0,1,2\ldots\} \tag{4}$$

where $$\phi_n(\lambda,m)=\left(\frac{n+\lambda}{1+\lambda}\right)^m \tag{5}$$

Special cases of this operator include the Uraegaddi and Somanatha operator in the case $\lambda=1$, [19], and for $\lambda=0$, the operator $\Im_\lambda^m$ reduces to to well-known Salagean operator introduced by Salagean [14].

Inspired by the earlier works (see [1,2,3,7,8,11,12,18]), we define the class as follows:

Definition 1.1. The function $f(z)$ of the form (1) is in the class $S_\lambda^m(\mu,\gamma,\varsigma)$ if it satisfies the inequality

$$\operatorname{Re}\left\{\frac{z\left(\Im_\lambda^m f(z)\right)'}{(1-\mu)z+\mu\Im_\lambda^m f(z)}-\gamma\right\}>\varsigma\left|\frac{z\left(\Im_\lambda^m f(z)\right)'}{(1-\mu)z+\mu\Im_\lambda^m f(z)}-1\right|$$

for $0\le\mu\le1$, $0\le\gamma\le1$ and $\varsigma\ge0$, $-1<\lambda\le1$, *and* $m\in N_0=\{0,1,2\ldots\}$

Further we define $TS_\lambda^m(\mu,\gamma,\varsigma)=S_\lambda^m(\mu,\gamma,\varsigma)\cap T$.

The aim of this paper is to study the coefficient bounds, radii of close-to-convex, starlikeness, and convex linear combinations for the functions in $TS_\lambda^m(\mu,\gamma,\varsigma)$.

2. Coefficient estimates

Theorem 2.1. A function *f(z)* of the form (1) is in $S_\lambda^m(\mu,\gamma,\varsigma)$

$$\sum_{n=2}^{\infty}[n(1+\varsigma)-\mu(\gamma+\varsigma)]\,\phi_n(\lambda,m)\left|a_n\right|\le1-\gamma \tag{6}$$

where $0\le\mu\le1$, $0\le\gamma\le1$, $\varsigma\ge0$ and $\phi_n(\lambda,m)$ is given by (5).

Proof: It suffices to show that

$$\varsigma\left|\frac{z\left(\Im_\lambda^m f(z)\right)'}{(1-\mu)z+\mu\Im_\lambda^m f(z)}-1\right|-\operatorname{Re}\left\{\frac{z\left(\Im_\lambda^m f(z)\right)'}{(1-\mu)z+\mu\Im_\lambda^m f(z)}-1\right\}\le1-\gamma$$

We have

$$\varsigma\left|\frac{z\left(\Im_\lambda^m f(z)\right)'}{(1-\mu)z+\mu\Im_\lambda^m f(z)}-1\right|-\operatorname{Re}\left\{\frac{z\left(\Im_\lambda^m f(z)\right)'}{(1-\mu)z+\mu\Im_\lambda^m f(z)}-1\right\}$$

$$\le(1+\varsigma)\left|\frac{z\left(\Im_\lambda^m f(z)\right)'}{(1-\mu)z+\mu\Im_\lambda^m f(z)}-1\right|$$

$$\le(1+\varsigma)\frac{\sum_{n=2}^{\infty}(n-\mu)\phi_n(\lambda,m)\left|a_n\right|\left|z\right|^{n-1}}{1-\sum_{n=2}^{\infty}\mu\phi_n(\lambda,m)\left|a_n\right|\left|z\right|^{n-1}}$$

$$\leq (1+\varsigma)\frac{\sum_{n=2}^{\infty}(n-\mu)\phi_n(\lambda,m)|a_n|}{1-\sum_{n=2}^{\infty}\mu\phi_n(\lambda,m)|a_n|}$$

The last expression is bounded above by $(1-\gamma)$ if

$$\sum_{n=2}^{\infty}[n(1+\varsigma)-\mu(\gamma+\varsigma)]\ \phi_n(\lambda,m)|a_n|\leq 1-\gamma$$

and the proof is complete.

Theorem 2.2. Let $0\leq\mu\leq 1$, $0\leq\gamma\leq 1$ and $\varsigma\geq 0$ then a function f of the form (1.3) to be in the class $TS_{\lambda}^{m}(\mu,\gamma,\varsigma)$ if and only if

$$\sum_{n=2}^{\infty}[n(1+\varsigma)-\mu(\gamma+\varsigma)]\ \phi_n(\lambda,m)\leq 1-\gamma \tag{7}$$

where $\phi_n(\lambda,m)$ are given by (5)

Proof: In view of Theorem 2.1, we need only to prove the necessity. If $f\in TS_{\lambda}^{m}(\mu,\gamma,\varsigma)$ and z is real then

$$\operatorname{Re}\left\{\frac{1-\sum_{n=2}^{\infty}n\phi_n(\lambda,m)a_n z^{n-1}}{1-\sum_{n=2}^{\infty}\mu\phi_n(\lambda,m)a_n z^{n-1}}-\gamma\right\}>\varsigma\left|\frac{\sum_{n=2}^{\infty}(n-\mu)\phi_n(\lambda,m)a_n z^{n-1}}{1-\sum_{n=2}^{\infty}\mu\phi_n(\lambda,m)a_n z^{n-1}}\right|$$

Letting $z\to 1$ along the real axis, we obtain the desired inequality

$$\sum_{n=2}^{\infty}[n(1+\varsigma)-\mu(\gamma+\varsigma)]\ \phi_n(\lambda,m)|a_n|\leq 1-\gamma,$$

where $0\leq\mu<1$, $0\leq\gamma\leq 1$ $\varsigma\geq 0$ and $\phi_n(\lambda,m)$ are given by (5).

Corollary 2.1. If $f(z)\in TS_{\lambda}^{m}(\mu,\gamma,\varsigma)$, then

$$|a_n|\leq\frac{1-\gamma}{[n(1+\varsigma)-\mu(\gamma+\varsigma)]\phi_n(\lambda,m)} \tag{8}$$

where $0\leq\mu<1$, $0\leq\gamma\leq 1$ $\varsigma\geq 0$ and $\phi_n(\lambda,m)$ are given by (5). Equality holds for the function

$$f(z)=z-\frac{1-\gamma}{[n(1+\varsigma)-\mu(\gamma+\varsigma)]\phi_n(\lambda,m)}z^n. \tag{9}$$

Theorem 2.3. Let $f_1(z) = z$ and

$$f_n(z) = z - \frac{1-\gamma}{[n(1+\varsigma) - \mu(\gamma+\varsigma)]\phi_n(\lambda,m)} z^n, \quad n \geq 2. \tag{10}$$

Then *f(z)* ∈ $TS_\lambda^m(\mu,\gamma,\varsigma)$, if and only if it can be expressed in the form

$$f(z) = \sum_{n=1}^{\infty} w_n f_n(z), \; w_n \geq 0, \sum_{n=1}^{\infty} w_n = 1 . \tag{11}$$

Proof: Suppose *f(z)* can be written as in (11). Then

$$f(z) = z - \sum_{n=2}^{\infty} w_n \frac{1-\gamma}{[n(1+\varsigma) - \mu(\gamma+\varsigma)]\phi_n(\lambda,m)} z^n .$$

Now, $\sum_{n=2}^{\infty} w_n \frac{(1-\gamma)[n(1+\varsigma) - \mu(\gamma+1)]\phi_n(\lambda,m)}{(1-\gamma)[n(1+\varsigma) - \mu(\gamma+1)]\phi_n(\lambda,m)} = \sum_{n=2}^{\infty} w_n = 1 - w_1 \leq 1.$

Thus *f(z)* ∈ $TS_\lambda^m(\mu,\gamma,\varsigma)$.

Conversely, let us have *f(z)* ∈ $TS_\lambda^m(\mu,\gamma,\varsigma)$.Then by using (8), we get

$$w_n = \frac{[n(1+\varsigma) - \mu(\gamma+1)]\phi_n(\lambda,m)}{(1-\gamma)} a_n \;, \; n \geq 2$$

and $w_1 = 1 - \sum_{n=2}^{\infty} w_n$. Then we have $f(z) = \sum_{n=1}^{\infty} w_n f_n(z)$ and hence this completes the proof of Theorem.

Theorem 2.4. The class $TS_\lambda^m(\mu,\gamma,\varsigma)$ is a convex set.

Proof: Let the function

$$f_j(z) = z - \sum_{n=2}^{\infty} a_{n,j} z^n \;, \quad a_{n,j} \geq 0, \text{ j = 1,2} \tag{12}$$

be in the class $TS_\lambda^m(\mu,\gamma,\varsigma)$. It sufficient to show that the function $h(z)$ defined by

$$h(z) = \xi f_1(z) + (1-\xi) f_2(z) \;, \; 0 \leq \xi < 1,$$

is in the class $TS_\lambda^m(\mu,\gamma,\varsigma)$. Since

$$h(z) = z - \sum_{n=2}^{\infty} \left[\xi a_{n,1} + (1-\xi) a_{n,2}\right] z^n \;,$$

An easy compution with the aid of of Theorem 2.2, gives

$$\sum_{n=2}^{\infty} [n(1+\varsigma) - \mu(\gamma+\varsigma)] \xi \phi_n(\lambda,m) a_{n,1} + \sum_{n=2}^{\infty} [n(1+\varsigma) - \mu(\gamma+\varsigma)] \, (1-\xi)\phi_n(\lambda,m) a_{n,2}$$
$$\leq \xi(1-\gamma) + (1-\xi)(1-\gamma)$$
$$\leq (1-\gamma),$$

which implies that $h \in TS_\lambda^m(\mu,\gamma,\varsigma)$.

Hence $TS_{\lambda}^{m}(\mu,\gamma,\varsigma)$ is convex.

3. Radii of close-to-convexity and starlikeness

Next we obtain the radii of close–to-convexity, starlikeness and convexity for the class $TS_{\lambda}^{m}(\mu,\gamma,\varsigma)$.

Theorem 3.1. Let the function $f(z)$ defined by (3) belong to the class $TS_{\lambda}^{m}(\mu,\gamma,\varsigma)$ Then $f(z)$

is close-to-convex of order δ $(0\le\delta<1)$ in the disc $|z|<r_1$, where

$$r_1=\inf_{n\ge2}\left[\frac{(1-\delta)\sum_{n=2}^{\infty}[n(1+\varsigma)-\mu(\gamma+\varsigma)]\phi_n(\lambda,m)}{n(1-\gamma)}\right]^{1/n-1}, \quad n\ge2. \tag{13}$$

The result is sharp, with the extremal function $f(z)$ is given by (10)

Proof: Given $f\in T$, and f is close-to-convex of order δ, we have

$$|f'(z)-1|<1-\delta \tag{14}$$

For the left hand side of (14) we have

$$|f'(z)-1|\le\sum_{n=2}^{\infty}na_n|z|^{n-1}$$

The last expression is less than $1-\delta$

$$\sum_{n=2}^{\infty}\frac{n}{1-\delta}a_n|z|^{n-1}\le1.$$

Using the fact, that *f(z)* ∈ $TS_{\lambda}^{m}(\mu,\gamma,\varsigma)$ if and only if

$$\sum_{n=2}^{\infty}\frac{[n(1+\varsigma)-\mu(\gamma+\varsigma)]\phi_n(\lambda,m)}{(1-\gamma)}a_n\le1,$$

We can (14) is true if

$$\frac{n}{1-\delta}|z|^{n-1}\le\frac{[n(1+\varsigma)-\mu(\gamma+\varsigma)]\phi_n(\lambda,m)}{(1-\gamma)}$$

or, equivalently,

$$|z|\le\left\{\frac{(1-\delta)[n(1+\varsigma)-\mu(\gamma+\varsigma)]\phi_n(\lambda,m)}{n(1-\gamma)}\right\}^{1/n-1}$$

which completes the proof.

Theorem 3.2. Let the function $f(z)$ defined by (3) belong to the class $TS_{\lambda}^{m}(\mu,\gamma,\varsigma)$ Then $f(z)$

is starlike of order of order δ $(0 \le \delta < 1)$ in the disc $|z| < r_2$, where

$$r_2 = \inf_{n\ge 2}\left[\frac{(1-\delta)\sum_{n=2}^{\infty}[n(1+\varsigma)-\mu(\gamma+\varsigma)]\phi_n(\lambda,m)}{(n-\delta)(1-\gamma)}\right]^{1/n-1} \qquad (15)$$

The result is sharp, with extremal function $f(z)$ is given by (2.5).

Proof: Given $f \in T$, and f is starlike of order δ, we have

$$\left|\frac{zf'(z)}{f(z)}-1\right| < 1-\delta \qquad (16)$$

For the left hand side of (16), we have

$$\left|\frac{zf'(z)}{f(z)}-1\right| \le \sum_{n=2}^{\infty}\frac{(n-1)a_n|z|^{n-1}}{1-\sum_{n=2}^{\infty}a_n|z|^{n-1}}$$

The last expression is less than

$\sum_{n=2}^{\infty}\frac{n-\delta}{1-\delta}a_n|z|^{n-1} < 1$.Using the fact that $f(z) \in TS_{\lambda}^{m}(\mu,\gamma,\varsigma)$ if and if

$$\sum_{n=2}^{\infty}\frac{[n(1+\varsigma)-\mu(\gamma+\varsigma)]\phi_n(\lambda,m)}{(1-\gamma)}a_n \le 1,$$

We can say (16) is true if

$$\sum_{n=2}^{\infty}\frac{n-\delta}{1-\delta}|z|^{n-1} \le \frac{[n(1+\varsigma)-\mu(\gamma+\varsigma)]\phi_n(\lambda,m)}{(1-\gamma)}$$

or equilently

$$|z|^{n-1} \le \frac{(1-\delta)[n(1+\varsigma)-\mu(\gamma+\varsigma)]\phi_n(\lambda,m)}{(n-\delta)(1-\gamma)}$$

which yields the starlikeness of the family.

4. Integral means inequalities

In [16], Silverman found that the function $f_2(z) = z-\frac{z^2}{2}$ is often extremal over the family T. He applied this function to resolve his integral means inequality conjectured [16] and settled in [17], that

$$\int_0^{2\pi}\left|f(re^{i\varphi})\right|^{\eta}d\varphi \le \int_0^{2\pi}\left|f_2(re^{i\varphi})^{\eta}\right| d\varphi\ ,$$

for all $f \in T$, $\eta > 0$ and $0 < r < 1$. In [16] , he also proved his conjecture for the subclasses $T^*(\alpha)$ and $C(\alpha)$ of T.

Now, we prove Silverman ‘s conjecture for the class of functions $TS_\lambda^m(\mu,\gamma,\varsigma)$.

We need the concept of subordination between analytic functions and a subordination theorem of Littlewood [9].

Two functions f and g, which are analytic in E, the function f is said to be subordinate to g in E if there exists a function w analytic in E with

$w(0)=0$, $|w(z)|<1$, $(z\in E)$ Such that $f(z)=g(w(z))$, $(z\in E)$.

We denote this subordination by $f(z)\prec g(z)$. ($\prec$ denotes subordination).

Lemma 4.1. If the functions f and g are analytic in E with $f(z)\prec g(z)$, then for $\eta>0$ and $z=re^{i\varphi}$ $0<r<1$,

$$\int_0^{2\pi}\left|g(re^{i\varphi})\right|^{\eta}d\varphi\leq\int_0^{2\pi}\left|f(re^{i\varphi})\right|^{\eta}d\varphi$$

Now, we discuss the integral means inequalities for functions f in $TS_\lambda^m(\mu,\gamma,\varsigma)$

$$\int_0^{2\pi}\left|g(re^{i\varphi})\right|^{\eta}d\varphi\leq\int_0^{2\pi}\left|f(re^{i\varphi})\right|^{\eta}d\varphi$$

Theorem 4.1. Let $f\in TS_\lambda^m(\mu,\gamma,\varsigma)$, $0\leq\mu<1$, $0\leq\gamma\leq1$, and $f_2(z)$ be defined by

$$f_2(z)=z-\frac{1-\gamma}{\varphi_2(\lambda,m,\mu,\varsigma,\gamma)}z^2 \tag{17}$$

Proof: For $f(z)=z-\sum_{n=2}^{\infty}a_nz^n$, (17) is equivalent to

$$\int_0^{2\pi}\left|1-\sum_{n=2}^{\infty}a_nz^{n-1}\right|^{\eta}d\varphi\leq\int_0^{2\pi}\left|1-\frac{1-\gamma}{\varphi_2(\lambda,m,\mu,\varsigma,\gamma)}z\right|^{\eta}d\varphi$$

By Lemma 4.1, it is enough to prove that

$$1-\sum_{n=2}^{\infty}a_nz^{n-1}\prec1-\frac{1-\gamma}{\varphi_2(\lambda,m,\mu,\varsigma,\gamma)}z$$

Assuming

$$1-\sum_{n=2}^{\infty}a_nz^{n-1}\prec1-\frac{1-\gamma}{\varphi_2(\lambda,m,\mu,\varsigma,\gamma)}w(z),$$

and using (7), we obtain

$$|w(z)|=\left|\sum_{n=2}^{\infty}\frac{\varphi_2(\lambda,m,\mu,\varsigma,\gamma)}{1-\gamma}a_nz^{n-1}\right|\leq|z|\sum_{n=2}^{\infty}\frac{\varphi_2(\lambda,m,\mu,\varsigma,\gamma)}{1-\gamma}a_n\leq|z|$$

where $\varphi_n(\lambda,m,\mu,\varsigma,\gamma)=[n(1+\varsigma)-\mu(\gamma+\varsigma)]\phi_n(\lambda,m)$

This completes the proof.

5. Conclusion

Motivated by this approach in the present study we introduced a new subclass of analytic functions involving multiplier transformations obtained necessary and sufficient conditions for this Classes. Furthermore radii of close-to-convexity and starlikeness and Integral means inequality results are obtained and therefore it may be considered as a useful tool for those who are interested in the above mention topics for the research.

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