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Brief Communication

The Minimum 1-Degree and Hamiltonicity of Triangle-Free Graphs

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Abstract. Let G be a graph. For a vertex u in G, its 1-degree is defined as the average of the degrees of all the vertices which are adjacent to u. The minimum 1-degree of a graph G is defined as the smallest 1-degree among all the 1-degrees of vertices in G. A graph G is triangle-free if no three vertices in G can form a complete graph of order three. In this short note, we present a minimum 1-degree condition for a triangle-free graph to be Hamiltonian.

Keywords: The minimum 1-degree, Hamiltonian graph, triangle-free graph

AMS Mathematics Subject Classification (2010): 05C45

1. Introduction

We consider only finite undirected graphs without loops or multiple edges. Notation and terminology not defined here follow those in [1]. Let G = (V, E) be a graph with n vertices and e edges. For a vertex u in G, we use N(u) to denote all the vertices in G which are adjacent to u. The degree of a vertex u, denoted $d_G(u)$, is defined as $|N(u)|$. We use δ_G to denote the minimum degree of G. The distance between two vertices u and v, denoted $d_G(u, v)$, in G is defined as the number of edges in the shortest path joining u and v. For a vertex u in G, we define D(G, u, k) as $\{x : x \in V, d_G(u, x) = k\}$, where k is a non-negative integer. The k-degree of u, denoted $d_G(u, k)$, in G is defined as $\sum_{x \in D(G, u, k)} d(x)/|D(G, u, k)|$. The minimum k-degree, denoted $\delta_G(k)$, of G is defined as $\min_{u \in V} d_G(u, k)$. Obviously, $d_G(u) = d_G(u, 0)$, $\delta_G = \delta_G(0)$, and $d_G(u, 1) = \sum_{x \in N(u)} d(x)/d(u)$ provided $d(u) \neq 0$. A set of vertices in a graph G is independent if the vertices in the set are pairwise nonadjacent. The join of two disjoint graphs G_1 and G_2 is a graph formed from G_1 and G_2 by joining every vertex of G_1 to every vertex of G_2. A graph G is triangle-free if no three vertices in G can form a complete graph of order three. A cycle C in a graph G is called a Hamiltonian cycle of G if C contains all the vertices of G. A graph G is called Hamiltonian if G has a Hamiltonian cycle.

In this short note, we present the following theorem in which a minimum 1-degree condition for a triangle-free graph to be Hamiltonian is obtained.

Theorem 1.1. Let G be a 2-connected triangle-free graph with $n \geq 3$ vertices and e edges. If $\delta_G(1) \geq n/2$, then G is Hamiltonian.

2. A Lemma

The following result which is Theorem 5 proved by Zhou in [2] plays a vital role in our proof of Theorem 1.1 in Section 3.

Lemma 2.1. Let G = (V, E) be a triangle-free graph with n vertices and e > 0 edges. Then $\sum_{u \in V} d^2(u) \leq ne$ and equality holds if and only if G is a complete bipartite graph.

3. Proofs and remarks

Proof of Theorem 1.1. Suppose G is a graph satisfying the conditions in Theorem 1.1. Then e > 0. Since $\delta_G(1) \geq n/2$, we, for each vertex u in G, have that $\sum_{x \in N(u)} d(x)/d(u) = d_G(u, 1) \geq \delta_G(1) \geq n/2$. Thus $\sum_{x \in N(u)} d(x) \geq nd(u)/2$. Therefore

$$\sum_{u \in V} d^2(u) = \sum_{u \in V}\sum_{x \in N(u)} d(x) \geq \sum_{u \in V} (n\, d(u))/2 = (n \sum_{u \in V} d(u))/2 = ne.$$

Lemma 2.1 implies that $\sum_{u \in V} d^2(u) = ne$. So, we, by Lemma 2.1 again, have that G is a complete bipartite graph. Suppose G is $K_{r, s}$. Then $\delta_G(1) = \min\{s, t\}$. If $s \neq t$, then $\delta_G(1) \leq (n - 1)/2$, a contradiction. Thus s = t and G is Hamiltonian. ■

Remark 3.1. Let G be a complete bipartite graph of $K_{r, r+1}$ with $r \geq 2$. Obviously, G is a 2-connected triangle-free graph with $\delta_G(1) \geq (n - 1)/2$. Note that G is not Hamiltonian. Thus the condition of $\delta_G(1) \geq n/2$ in Theorem 1.1 cannot be relaxed.

Remark 3.2. Let K_r be a complete graph of order r and its vertex set is $\{y_1, y_2, ..., y_r\}$, where $r \geq 2$. Let H be a graph which is a join between K_r and a graph consisting of r independent vertices, where $r \geq 2$. Define a graph G = (V, E) as follows. $V := V(H) \cup \{z\}$ and $E := E(H) \cup \{zy_1, zy_2, ..., zy_r\}$, where $r \geq 2$ and z is a vertex which is not equal to any vertex in V(H). Obviously, G is a 2-connected graph with $\delta_G(1) \geq n/2$. Note that G is not Hamiltonian. Thus the condition that G is triangle-free in Theorem 1.1 cannot be dropped.

4. Conclusion

In this note, we present a sufficient condition based upon the minimum 1-degree for triangle-free graphs to be Hamiltonian.

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REFERENCES

1. J.A. Bondy and U.S.R. Murty, *Graph Theory with Applications*, Macmillan, London and Elsevier, New York (1976).
2. B. Zhou, Zagreb Indices, *MATCH Commun Math Comput Chem.*. 52 (2004) 113-118.