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Finitely Generated n-Ideals of a NearLattice

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Abstract. By a near lattice we mean a meet semi lattice with the property that any two elements possessing a common upper bound, have a supremum. For a near lattice S, if n is neutral and upper then the set of all finitely generated n-ideals, $F_n(S)$ is a lattice and the set of all principal n-ideals, $P_n(S)$ is again a nearlattice. In this paper, we have proved that when n is an upper element of a distributive nearlattice S, then $F_n(S)$ is generalized Boolean if and only if $P_n(S)$ is semi Boolean. Moreover we have also shown that $F_n(S)$ is generalized Boolean if and only if the set of all prime n-idals P(S) is unordered by set inclusion, when n is an upper and S is distributive.

Keywords: Near lattice, Finitely generated n-ideal, Principal n-ideal, Semi Boolean near lattice, Prime n-ideal.

AMS Mathematics Subject Classifications (2010): 06A12, 06A99, 06B10.

1. Introduction

A *Nearlattice* S is a *meet semilattice* with the property that any two elements possessing a common upper bound, have a supremum. This property is known as the upper bound property. Nearlattice S is called a *distributive* nearlattice if for all $a,b,c \in S$, $a \wedge (b \vee c) = (a \wedge b) \vee (a \wedge c)$ provided $b \vee c$ exists. A near lattice S is called a *medial near lattice* if for all $x, y, z \in S$,
$m(x, y, z) = (x \wedge y) \vee (y \wedge z) \vee (z \wedge x)$ exists.

Let n be a fixed element of S. A convex sub nearlattice containing n is called an n-*ideal*. An element $n \in S$ is called a *medial* element if $m(x, n, y) = (x \wedge y) \vee (x \wedge n) \vee (y \wedge n)$ exists for all $x, y \in S$. Element n is called an *upper* element if $x \vee n$ exists for all $x \in S$. Of course, every upper element is medial.

An element n of a nearlattice S is called a *standard* element if for all t, x, y$\in$S, $t\wedge[(x\wedge y)\vee(x\wedge n)]=(t\wedge x\wedge y)\vee(t\wedge x\wedge n)$. Moreover, n is called a *neutral* element if (i) n is standard, and (ii) $n\wedge[(t\wedge x)\vee(t\wedge y)]=(n\wedge t\wedge x)\vee(n\wedge t\wedge y)$ for all t, x, y$\in$S. The set of all n-ideals of S is denoted by $I_n(S)$, which is an algebraic lattice. An n-ideal generated by a finite numbers of elements $a_1,a_2,...,a_m$ is called a *finitely generated* n-*ideal* and is denoted by $<a_1,a_2,...,a_m>_n$. The set of all finitely generated n-ideals is denoted by $F_n(S)$. By [1], we know that $F_n(S)$ is a lattice. The n-ideal generated by a single element x, is called a *principal* n-*ideal* and is denoted by $<x>_n$. The set of all principal n ideals is denoted by $P_n(S)$. If n is medial and standard, then it is well known that $<x>_n\cap<y>_n=<m(x,n,y)>_n$. Thus, when n is medial and standard, $P_n(S)$ is a meet semi lattice. We also know from [2] that when n is upper and medial, then $P_n(S)$ is also a nearlattice. When n is an upper element, $<x>_n$ is the interval $[x\wedge n, x\vee n]$. If S is a lattice, then $<a_1,a_2,....,a_m>_n=[a_1\wedge a_2\wedge\ldots\wedge a_m\wedge n, a_1\vee a_2\vee\ldots\vee a_m\vee n]$. Thus the members of $F_n(S)$ are of the form [a, b], $a\le n\le b$. Moreover, $[a, b]\cap[c, d]=[a\vee c, b\wedge d]$ and $[a, b]\vee[c, d]=[a\wedge c, b\vee d]$. an n-ideal P is called a *prime* n-*ideal* if for all a, b$\in$L ,m(a, n, b) $\in$P implies either a$\in$P or b$\in$P. For detailed literature on nearlattices and the description of n-ideals we refer the reader to consul [2,3,4,5].
We start with the following result given in [6, Corollary 1.4.5].

Lemma 1. *Let* n *be an upper and neutral element of* **S.** *Then any finitely generated* n-*ideal contained in a principal n-ideal is principal.*

A nearlattice S with 0 is *sectionally complemented* if the interval [0, x] is complemented for each x$\in$S.

Of course, every relatively complemented nearlattice S with 0 is sectionally complemented.

A nearlatice (lattice) S with 0 is called *semi Boolean(Generalized Boolean)* if it is distributive and the interval [0, x] is complemented for each x$\in$S.
An element $n\in S$ is called a *central element* if it is upper, neutral and complemented in each interval containing it.

Following result is due to [3].

Theorem 2. *For a neutral element* n *of a nearlattice* S, n *is central if and only if* n *is upper and* $P_n(S)\cong(n]^d\times[n)$.

Following results are easy consequences of the above theorem.

Corollary 3. *Let* S *be a nearlattice and* n$\in$S *be a central element. Then* $P_n(S)$ *is sectionally complemented if and only if the intervals* [a, n] *and* [n, b] *are complemented for each* a, b$\in$S($a\le n\le b$).

We know that if n is medial in distributive nearlattice S, then $I_n(S)$ is also distributive and hence $P_n(S)$ (if it is nearlattice) is also distributive.

Corollary 4. *If* n *is central element of a distributive nearlattice* S, *then* $P_n(S)$ *is semiBoolean if and only if the intervals* [a, n] *and* [n, b] *are complemented for each* $a, b \in S (a \leq n \leq b)$.

Now we prove the following result when n is only an upper element rather than a central element .Thus it is an improvement of the above results.

Theorem 5. *Let* n *be an upper element of a distributive nearlattice* S. *Then the following conditions are equivalent.*
(i) $P_n(S)$ *is semi Boolean*.
(ii) [a, n] *and* [n, b] *are complemented for all* a< n< b.

Proof: (i) $\Rightarrow$(ii). Suppose $P_n(S)$ is semi Boolean and let $a \leq y \leq n$. Therefore, $\{n\} \subseteq <y>_n \subseteq <a>_n$ which implies $\{n\} \subseteq [y, n] \subseteq [a, n]$. Let $<t>_n$ be the relative complement of $<y>_n$ in [$\{n\}$, $<a>_n$]. Then $t \leq n$. Also, $<t>_n \cap <y>_n = \{n\}$ and $<t>_n \vee <y>_n = <a>_n$. Now $<t>_n \cap <y>_n = \{n\}$ implies $[t, n] \wedge [y, n] = \{n\}$ and so $[t \vee y, n] = \{n\}$ implies $t \vee y = n$. Also, $<t>_n \vee <y>_n = <a>_n$ implies $[t, n] \vee [y, n] = [a, n]$ and so $[t \wedge y, n] = [a, n]$. Thus $t \wedge y = a$. Hence, [a, n] is complemented. Similarly we can prove dually that [n, b] is also complemented.

(ii) $\Rightarrow$(i). Suppose [a, n] and [n, b] are complemented for all a< n< b. Consider $\{n\} \subseteq <p>_n \subseteq <q>_n$. Then $q \wedge n \leq p \wedge n \leq n \leq p \vee n \leq q \vee n$. Since $[n, q \vee n]$ is complemented, so there exists $s \in [n, q \vee n]$, such that $(p \vee n) \wedge s = n$ and $p \vee n \vee s = q \vee n$. Again as $[q \wedge n, n]$ is complemented, so there exists $r \in [q \wedge n, n]$ such that $r \wedge p \wedge n = q \vee n$ and $r \vee (p \wedge n) = n$. Then $[r, s] \cap <p>_n = \{n\}$ and $[r, s] \vee <p>_n = <q>_n$.

That is [r, s] is relative complement of $<p>_n$ in [$\{n\}$, $<q>_n$]. But by lemma 1, we know that any finitely generated n-ideal contained in a principal n-ideal is principal. Hence $[r, s] \in P_n(S)$ and $P_n(S)$ is semi Boolean.

Theorem 6. *Let* S *be a distributive nearlattice with an upper element* n. *Then the following conditions are equivalent.*

(i) $F_n(S)$ *is generalized Boolean.*

(ii) $P_n(S)$ *is semi Boolean.*

(iii) [a, n] *and* [n, b] *are complemented for all* a< n< b.

Proof: By theorem 5, it is sufficient to show **(i)** $\Leftrightarrow$ (ii)

(i) $\Rightarrow$(ii) is obvious by lemma 1. Conversely, let (ii) holds. Suppose $\{n\} \subseteq < x_1, x_2, ..., x_p >_n \subseteq < y_1, y_2, ..., y_s >_n$.

That is, $\{n\} \subseteq < x_1, x_2, ..., x_p >_n \cap < y_1 >_n \subseteq < y_1 >_n$, which implies $\{n\} \subseteq (< x_1 >_n \vee < x_2 >_n \vee .... \vee < x_p >_n) \cap < y_1 >_n \subseteq < y_1 >_n$ and so $\{n\} \subseteq [(< x_1 >_n \cap < y_1 >_n) \vee .... \vee (< x_p >_n \cap < y_1 >_n) \subseteq < y_1 >_n$.

Thus, $\{n\} \subseteq < m(x_1, n, y_1 >_n \vee .... \vee < m(x_p, n, y_1 >_n \subseteq < y_1 >_n$. By lemma 1, $< m(x_1, n, y_1 >_n \vee .... \vee < m(x_p, n, y_1 >_n$ is a principal n-ideal. Now let, $< t_1 >_n = < m(x_1, n, y_1 >_n \vee .... < m(x_p, n, y_1 >_n$ and let $< r_1 >_n$ be a complement of $< t_1 >_n$ such that $< r_1 >_n \vee < t_1 >_n = < y_1 >_n$ and $< r_1 >_n \cap < t_1 >_n = \{n\}$. So we can get, $< t_i >_n$; i = 1,2,...,s and the complement $< r_i >_n$; i = 1,2,...,s of $< t_i >_n$ such that $< r_1, r_2, ..., r_s >_n \vee < x_1, x_2, ..., x_p >_n = < y_1, y_2, ..., y_s >_n$ and $< r_1, r_2, ..., r_s >_n \cap < x_1, x_2, ..., x_p >_n$

$= (< r_1 >_n \cap < x_1, x_2, ..., x_p >_n) \vee .... \vee (< r_s >_n \cap < x_1, x_2, ..., x_p >_n)$

$= \{n\} \vee \{n\} \vee .... \vee \{n\} = \{n\}$.Therefore, $F_n(S)$ is generalized Boolean.

Following results are due to [5]. These will be needed for further development of the thesis.

Lemma 7. *If* S_1 *is a subnearlattice of a distributive nearlattice* S *and* P_1 *is a prime ideal(filter) in* S_1, *then there exists a prime ideal* P *in* S *such that* $P_1 = P \cap S_1$.

In lattice theory, it is well known that a distributive lattice L with 0 and 1 is Boolean if and only if its set of prime ideals is unordered by set inclusion. Following result due to [5] have generalized this result for distributive nearlattices with 0.

Theorem 8. *If* S *is a distributive nearlattice with* 0, *then* S *is semiBoolean if and only if its set of prime ideals (filters) is unordered by set inclusion.*

We conclude the paper by the generalization of above result for n-ideals.

Theorem 9. *Let* S *be a distributive nearlattice* $n \in S$ *be an upper element. Then the following conditions are equivalent.*

(i) $F_n(S)$ *is generalized Boolean*.

(ii) *The set of prime* n-*ideals* P(S) *of* S *is unordered by set inclusion*.

Proof: (i) $\Leftrightarrow$ (ii). Suppose $F_n(S)$ is generalized Boolean. Then by theorem 5 and theorem 6, the interval [x, n] and [n, y] are complemented for each x, y $\in$S with $x \le n \le y$. If P(S) is not unordered, suppose there are prime n-ideals P,Q with $P \subset Q$. Let $b \in Q - P$. Now, as Q is prime, there exists $a \in S$ such that $a \notin Q$. Then either $a \wedge n \notin Q$ or $a \vee n \notin Q$ (here $a \vee n$ exists an n is upper). For, otherwise $a \in Q$ by convexity.

Suppose $a \vee n \notin Q$, Since $[n, a \vee n]$ is complemented and $n \leq (a \wedge b) \vee n \leq a \vee n$, so there exists $t \in [n, a \vee n]$ such that $t \wedge [(a \wedge b) \vee n] = n$ and $t \vee [(a \wedge b) \vee n] = a \vee n$.

Since $t \wedge [(a \wedge b) \vee n] = m(t, n, (a \wedge b) \vee n) = n$, thus $t \wedge [(a \wedge b) \vee n] = m(t, n, (a \wedge b) \vee n) \in P$. Since P is prime, so either $t \in P$ or $(a \wedge b) \vee n \in P$. Now $n \leq (a \wedge b) \vee n \leq b \vee n$ implies $(a \wedge b) \vee n \in Q$. If $t \in P$, then $t \in Q$ and so $a \vee n = t \vee [(a \wedge b) \vee n] \in Q$, which gives a contradiction .

If $(a \wedge b) \vee n \in P$, then $(a \wedge b) \vee n = m(a \vee n, n, b) \in P$ implies $b \in P$ which is again a contradiction. Therefore, $a \vee n \in Q$.

Now if $a \wedge n \notin Q$, then $a \wedge b \wedge n \notin Q$ as $n \in Q$ and Q is convex. Since $b \wedge n$ has relative complement in $[a \wedge b \wedge n, n]$. Proceeding as above, again we arrive at a contradiction .Thus $a \wedge n \in Q$. Since both $a \wedge n$ and $a \vee n$ belong Q, so by convexity $a \in Q$. This gives a contradiction. Therefore the set of prime n-deals P(S) is unordered.

(ii) $\Rightarrow$ (i). Suppose that P(S) is unordered. Consider any interval [n, b] in S. Let P_1, Q_1 be two prime ideals of [n, b]. Then by lemma 7, there exist prime ideals P and Q of S such that $P_1 = P \cap [n, b]$ and $Q_1 = Q \cap [n, b]$.

Since P and Q contains n, so by [6, lemma 2.1.3], they are prime n-deals. Since P(S) is unordered, so P and Q are incomparable. This follows that P_1 and Q_1 are also incomparable.

If not, let $P_1 \subset Q_1$. Then for any $z \in P$, $(z \vee n) \wedge b \in [n, b]$ and $n \leq (z \vee n) \wedge b \leq z \vee n$ implies, $(z \vee n) \wedge b \in P_1 \subset Q_1$. Thus $(z \vee n) \wedge b \in Q$. But $b \notin Q$ as Q_1 is prime in [n, b]. Therefore, $z \vee n \in Q$ as Q is a prime ideal of S and so $z \in Q$. Hence $P \subset Q$, which is a contradiction .Therefore, by [1,Th.22, p-46], [n, b] is complemented.

Again, consider the interval [a, n]. Since the prime filters are the complements of prime ideals, so considering two prime filters of [a, n] and using the same argument as above we see that [a, n] is also complemented. Hence by theorem 5, P(S) is semi Boolean and by theorem 6, $F_n(S)$ is generalized Boolean.

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