Annals of Pure and Applied Mathematics
Vol. 2, No. 2, 2012, 107-113
ISSN: 2279-087X (P), 2279-0888(online)
Published on 18 December 2012
www.researchmathsci.org

Annals of
Pure and Applied Mathematics

On the Lattice of Weakly Induced $T_1 - L$ Topologies

Raji George[1] and T.P.Johnson[2]
[1]Department of Mathematics, St.Peter's College,
Kolenchery -682311,Ernakulam Dt ,Kerala State,India
E-mail:bobyraji@gmail.com
,
[2]Applied Sciences and Humanities Division,School of Engineering,
Cochin University of Science and Technology, Cochin-22,
Kerala State, India
E-mail:tpjohnson@cusat.ac.in

Received 2 December 2012; accepted 17 December 2012

Abstract. We investigate the lattice structure of the set $W_{1\tau}$ of all weakly induced $T_1 - L$ topologies defined by families of (completely) scott continuous functions on X. It is proved that this lattice is complete,not atomic,not distributive,not complemented and not dually atomic.From this we deduce the properties of the lattice $W_1(X)$ of all weakly induced $T_1 - L$ topologies on a given set X .

Keywords: Scott topology, weakly induced $T_1 - L$ topology, induced $T_1 - L$ topology, complete lattice, atom, dual atom.

AMS Mathematics Subject Classification (2010): 54A40

1. Introduction

The concept of induced fuzzy topological space was introduced by Weiss [14] .Lowen called these spaces a topologically generated spaces.Martin[9] introduced a generalised concept,weakly induced spaces,which was called semi induced space by Mashhour et al.[10].The notion of lower semicontinuous functions plays an important tool in defining the above concepts.In [5] Aygun et al. introduced a new class of functions from a topological space (X, τ) to a fuzzy lattice L with its scott topology called (completely) scott continuous functions as a generalisation of (completely) lower-semi continuous functions from (X, τ) to $[0,1]$.It is known that [6] lattice of L-topologies is complete,atomic and not complemented.In [7] Jose and Johnson genralised weakly induced spaces introduced by Martin[9] using the tool (completely) scott continuous functions and studied the lattice structure of the set $W(X)$ of all weakly induced L-topologies on a given set X .A related problem is to find subfamilies in $W(X)$ having certain properties.The collection of all weakly induced $T_1 - L$ topologies $W_1(X)$ form a lattice with natural order of set inclusion.In [12] Liu determined dual atoms in the lattice of T_1 topologies and Frolich [2] proved this lattice is dually atomic.Here we study properties of the lattice $W_{1\tau}$ of weakly induced $T_1 - L$ topologies defined by families of (completely) scott continuous functions with reference to τ on X .It has dual atoms if and only if the membership lattice L has dual atoms and it is not dually atomic in general.From the lattice $W_{1\tau}$ we deduce the lattice $W_1(X)$.

2. Preliminaries

Let X be a nonempty ordinary set and $L = L(\leq,\vee,\wedge,')$ be a complete completely distributive lattice with smallest element 0 and largest element $1, 0 \neq 1$ and with an order reversing involution $a \to a^{'} (a \in L)$ .We identify the constant function from X to L with value α by $\underline{\alpha}$ .The fundamental definition of L-fuzzy set theory and L-topology are assumed to be familiar to the leader in the sense of Chang[1].

Definition 2.1. *[11] A fuzzy point x_λ in a set X is a fuzzy set in X defined by* $x_\lambda(y) = \begin{cases} \lambda \text{ if } y = x \\ 0 \text{ if } y \neq x \end{cases}$ *where* $0 < \lambda \leq 1$

Definition 2.2. *[11] An L-topological space (X, F) is said to be a $T_1 - L$ topological space if for every two distinct fuzzy points x_p and y_q , with distinct support ,there exists an $f \in F$ such that $x_p \in f$ and $y_q \notin f$ and another $g \in F$ such that $y_q \in g$ and $x_p \notin g, \forall p, q \in L \setminus \{0\}$*

Remark 2.1. *We take the definition of fuzzy points $x_\lambda, 0 < \lambda \leq 1$ so as to include all crisp singletons.Hence every crisp T_1 topology is a $T_1 - L$ topology by identifying it with its characteristic function .If τ is any topology on a finite set, then τ is T_1, if and only if it is discrete.However,the same is not true in L-topology.*

Definition 2.3. *[4] An element $p \in L$ is called prime if $p \neq 1$ and whenever $a, b \in L$ with $a \wedge b \leq p$,then $a \leq p$ or $b \leq p$ .The set of all prime elements of L will be denoted by $P_r(L)$.*

Definition 2.4. *[13] The scott topology on L is the topology generated by the sets of the form $\{t \in L : t \quad p\}$, where $p \in P_r(L)$.Let (X, τ) be a topological space and $f : (X, \tau) \to L$ be a function where L has its scott topology ,we say that f is scott continuous if for every $p \in P_r(L), f^{-1}\{t \in L : t \quad p\} \in \tau$*

Remark 2.2. *When $L=[0,1]$,the scott topology coincides with the topology of topologically generated spaces of Lowen[8].The set*
$\omega_L(\tau)=\{f\in L^X;f:(X,\tau)\to L$ *is scott continuous}*
is an L-topology.It is the largest element in W_τ.If τ is a T_1 topology $\omega_L(\tau)$ is a T_1-L topology, we can denote it by $\omega_{1L}(\tau)$.An L-topology F on X is called an induced T_1-L topology if there exists a T_1 topology τ on X such that $F=\omega_{1L}(\tau)$.

Definition 2.5. *[5] Let (X,τ) be a topological space and $\alpha\in X$ .A function $f:(X,\tau)\to L$,where L has its scott topology , is said to be completely scott continuous at $\alpha\in X$ if for every $p\in P_r(L)$ with $f(\alpha)$ p ,there is a regular open neighbourhood U of α in (X,τ) such that $f(x)$ p for every $x\in U$ .That is $U\subset f^{-1}(\{t\in L:t\quad p\})$ and f is called completely scott continuous on X ,if f is completely scott continuous at every point of X .*

Note 1. Let F be a T_1-L topology on the set X , let F_c denote the $0-1$ valued members of F ,that is , F_c is the set of all characteristic mappings in F.Then F is a T_1-L topology on X .Define $F_c^*=\{A\subset X:\mu_A\in F_c$,where μ_A is the characteristic function of $A\}$.The T_1-L topological space (X,F_c) is same as the T_1 topological spaces (X,F_c^*)

Definition 2.6. *A T_1-L topological space (X,F) is said to be a weakly induced T_1-L topological space,if for each $f\in F,f$ is a scott continuous function from (X,F_c^*) to L.*

Definition 2.7. *If F is the collection of all scott continuous functions from (X,F_c^*) to L,then F is an induced space and $F=\omega_{1L}(F_c^*)$.*

Definition 2.8. *[15] An element of a lattice L is called an atom if it is the minimal element of $L\setminus\{0\}$.*

Definition 2.9. *[15] An element of a lattice L is called a dual atom if it is the maximal element of $L\setminus\{1\}$.*

Definition 2.10. *[15] A bounded lattice is said to be complemented if for all x in L*

there exists y in L such that $x \vee y = 1$ and $x \wedge y = 0$.

3. Lattice of weakly induced $T_1 - L$ topology

For a given T_1-topology on X , the family $W_{1\tau}$ of all weakly induced $T_1 - L$ topologies defined by families of scott continuous functions from (X, τ) to L forms a lattice under the natural order of set innclusion.The least upper bound of a collection of weakly induced $T_1 - L$ topologies belonging to $\omega_{1\tau}$ is the weakly induced $T_1 - L$ topology which is generated by their union and their greatest lower bound is their intersection.The smallest element is the crisp cofinite topology denoted by 0 and the largest element is $\omega_{1L}(\tau)$.Also for a T_1 topology τ on X ,the family $CW_{1\tau}$ of all weakly induced $T_1 - L$ topology defined by families of completely scott continuous functions from (X, τ) to L forms a lattice under the natural order of set inclusion.Since every completely scott continuous function is scott continuous,it follows that $CW_1\tau$ is a sublattice of $W_{1\tau}$.We note that $W_{1\tau}$ and $CW_{1\tau}$ coincide when each open set in τ is regular open.When $\tau = D$,the discrete topology on X ,these lattices coincide with lattice of weakly induced $T_1 - L$ topologies on X .

Theorem 3.1. *The lattice $W_{1\tau}$ is complete.*

Proof. Let S be a subset of $W_{1\tau}$ and let $G = \bigcap_{F \in S} F$ .Clearly G is a $T_1 - L$ topology.Let $g \in G$.Since each $F \in S$ is a weakly induced $T_1 - L$ topology,g is a scott continuous mapping from (X, F_c^*) to L. That is $g^{-1}(\{t \in L : t \quad p$, where $p \in P_r(L)\}) \in F_c^*$ for each $F \in S$ .Therefore $g^{-1}(\{t \in L ; t \quad p$ where $p \in P_r(L)\}) \in \bigcap_{F \in S} F_c^*$.Hence g is a scott continuous function from (X, G_c^*) to L ,where $(X, G_c^*) = (X, \bigcap_{F \in S} F_c^*)$.That is $G \in W_{1\tau}$ and G is the greatest lower bound of S .Let K be the set of upper bounds of S .Then K is nonempty since $1 = \omega_{1L}(\tau) \in K$.Using the above argument K has a greatest lower bound say H .Then this H is a least upper bound of S .Thus every subset S of $W_{1\tau}$ has greatest lower bound and least upper bound.Hence $W_{1\tau}$ is complete.

Note 2. Let CFT denote the crisp cofinite topology,where $CFT = \{\mu_A : A$ is a subset of X whose complement is finite $\} \cup \{\underline{0}\}$,where μ_A is the charcteristic function of A

Theorem 3.2. $W_{1\tau}$ *is not atomic.*

Proof. The atoms in $W_{1\tau}$ are $T_1 - L$ topologies generated by $CFT \cup \{x_\lambda\}, 0 < \lambda \le 1$ where x_λ is a fuzzy point.Let $\ell = \{f \in L^X : f(x) > 0$ for all but finite number of points of $X\} \cup \{0\}$.

$F_c = CFT \cup \{\underline{0}\}, F_c^*$ =discrete topology.Then ℓ is a weakly induced T_1 topology on X and it cannot be expressed as join of atoms.Hence $W_{1\tau}$ is not atomic.

Theorem 3.3. *[3] A lattice* L *is modular if and only if it has no sublattice isomorphic to* N_5 *,where* N_5 *is standard non modular lattice*

Theorem 3.4. $W_{1\tau}$ *is not distributive.*

Proof. Since every distributive lattice is necessarily modular,we prove that $W_{1\tau}$ is not modular. This can be illustrated with an example. Let $x_1, x_2, x_3 \in X, \alpha, \beta, \gamma \in (0,1)$ Let F be the weakly induced $T_1 - L$ topology generated by $CFT \cup \{f_1, f_2, f_3\}$ where f_1, f_2, f_3 are L-subsets defined by

$$f_1(y) = \begin{cases} \alpha \text{ when } y = x_1 \\ 0 \text{ when } y \neq x_1 \end{cases} \qquad f_2(y) = \begin{cases} \alpha \text{ when } y = x_1 \\ \beta \text{ when } y = x_2 \\ \gamma \text{ when } y = x_3 \\ 0 \text{ when } y \neq x_1, x_2, x_3 \end{cases}$$

$$f_3(y) = \begin{cases} \beta \text{ when } y = x_2 \\ \gamma \text{ when } y = x_3 \\ 0 \text{ when } y \neq x_2, x_3 \end{cases}$$

Let F_1 be the weakly induced $T_1 - L$ topology generated by $CFT \cup \{f_1\}$.

Let F_2 be the weakly induced $T_1 - L$ topology generated by $CFT \cup \{f_1, f_2\}$.

Let F_3 be the weakly induced $T_1 - L$ topology generated by $CFT \cup \{f_3\}$. Then $F_2 \vee F_3 = F$ and $F_1 \vee F_3 = F$ so that $\{CFT, F_1, F_2, F_3, F\}$ forms a sublattice of $W_{1\tau}$ isomorphic to N_5 ,where N_5 is the standard non modular lattice.Therefore $W_{1\tau}$ is not modular and hence not distributive.

Theorem 3.5. $W_{1\tau}$ *is not complemented.*

Proof. Let F be the weakly induced $T_1 - L$ topology generated by $CFT \cup \{x_\lambda\}$ .Then 1 is not a complement of F since $F \wedge 1 \neq 0$ .Let H be any weakly induced $T_1 - L$ topology other than 1.If $F \subset H$,then H cannot be the complement of F .If $F Ú H$, let $F \vee H = G$ and G has the subbasis $\{f \wedge h / f \in F, h \in H\}$ .Then G cannot be equal to $\omega_{1L}(\tau)$ .Hence it is not a complement of F .

Remark 3.1. *When* $\tau = D$ *,the discrete topology on* $X, W_{1D} = W_1(X)$ *,the collection of all weakly induced* L *-topologies on* X *.The family of all weakly induced* $T_1 - L$ *topologies is defined by scott continuous functions where each scott continuous function is a characteristic function, is a sublattice of* $W_1(X)$ *and is a lattice isomorphic to the lattice of all topologies on* X *.The elements of this lattice are called crisp* T_1 *topoloigies.*

Theorem 3.6. *The lattice of weakly induced* $L-$ *topologies* $W_1(X)$ *is not complemented.*

Proof. This follows from theorem 3.5

Theorem 3.7. *If* L *has dual atoms , then* $W_{1\tau}$ *has dual atoms.*

Proof. Let τ be a dual atom in the lattice of T_1 topologies. The only topology finer than τ is the discrete topology. Then there exists a subset A of X such that the simple expansion of τ by A is the discrete topology. Now consider $\omega_{1L}(\tau)$,the $T_1 - L$ topology consists of all scott continuous functions.Then the characteristic function μ_A of the subset A doesnot belong to $\omega_{1L}(\tau)$ .Then if α is a dual atom in L ,then the weakly induced $T_1 - L$ topology generated by $\omega_{1L}(\tau) \cup \mu_A^\alpha$ is a dual atom in $W_{1\tau}$ where

$$\mu_A^\alpha(x) = \begin{cases} \alpha \ if \ x \in A \\ 0 \ otherwise \end{cases}$$

Theorem 3.8. *If* L *has no dual atoms,then* W_{1L} *has no dual atoms.*

Proof. Let F be any weakly induced $T_1 - L$ topology other than $1 = \omega_{1L}(\tau)$ .Then we claim that there exists atleast one weakly induced $T_1 - L$ topology finer than F .Since F is a weakly induced $T_1 - L$ topology different from $\omega_{1L}(\tau), F$ cannnot contain all characteristic functions of subsets of X .Since L has no dual atoms,the collection S of L subsets not belonging to F is infinite. If $g \in S$,then $F(g)$, the simple expansion of F by g is a weakly induced $T_1 - L$ topology.Thus for any weakly induced $T_1 - L$ topology F, there exists a weakly induced $T_1 - L$ topology $G = F(g)$, such that $F \subset G \neq 1$. Hence the proof of the theorem is completed.

Comparing theorem 3.7 and Theorem 3.8 we have the following theorem.

Theorem 3.9. *The lattice of weakly induced* $T_1 - L$ *topologies* $W_{1\tau}$ *has dual atoms if and only if* L *has dual atoms.*

Theorem 3.10. $W_{1\tau}$ *is not dually atomic in general.*

Aknowledgement. The first author wishes to thank the University Grant Commission, India for giving financial suppport.

REFERENCES

1. Chang C.L., Fuzzy Topological Spaces, *J.Math.Anal.Appl.,* 24 (1968) 191-201.
2. Frolich, O., Das Halbordnungs System det.Topolegychen Raume aut einer Menge, *Math.Annalen,* 156 (1964) 79-95.
3. George Gratzer, University Of Manitoba, *Lattice Theory*.
4. Gierz, G., *A Compendium of Continuous Lattices*, Springer,Berlin (Dec. 9,1980).
5. Halis Aygum, Warner M.W. and Kudri S.R.T., Completely induced L-fuzzy topological spaces, *Fuzzy Sets and Systems*, 103 (1999)513-523.
6. Johnson T.P.: On Lattices Of L-topologies, *Indian Journal of Mathematics,* 46(1) (2004) 21-26.
7. Jose V.K. and Johnson T.P., Lattice of Weakly Induced topologies, *The Journal of Fuzzy Mathematics*, 16(1) (2008)327-333.
8. Lowen,R., Fuzzy Topological Spaces and Fuzzy Compactness, *J.Math.Anal.Appl.,*56 (1976) 621-633.
9. Martin H.W., On Weakly Induced Fuzzy Topological Spaces, *J.Math.Anal.Appl.*, 78 (1980) 634-637.
10. Mashhour A.S., Ghanim M.H. , Wakril A.E. and Moris N.M., Semiinduced Fuzzy Topolgies, *Fuzzy Sets and Systems*, 31 (1989) 1-18.
11. Rekha Srivastava, Lal S.N. and Srivastava A.K., On Fuzzy T_1 -topological Space, *J.Math.Anal.Appl.,* 127 (1987) 1-7.
12. Teng-Sun Liu, A Note On Maximal T_1 -topologies, *Portugal Math.,* 18 (1959) 235-236.
13. Warner M.W.and McLean R.G., On Compact Hausdorff L-fuzzy Spaces, *Fuzzy Sets and Systems*, 56 (1993) 103-110.
14. Weiss M.D., Fixed Points, Separation and Induced Topologies for Fuzzy Sets, *J.Math.Anal.Appl.*, 50 (1975) 142-150.
15. Ying-Ming Liu and Mao-Kang Luo, *Fuzzy Topology*, World Scienntific Publishing Company, River Edge, New Jersy, 1997.