Annals of Pure and Applied Mathematics
Vol. 2, No.2, 2012, 164-176
ISSN: 2279-087X (P), 2279-0888(online)
Published on 27 December 2012
www.researchmathsci.org

Annals of
Pure and Applied Mathematics

Two Dimensional Hénon Map with the Parameter Values 1 < *a* < 2, |*b*| < 1 in Dynamical Systems

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Received 17 December 2012; accepted 26 December 2012

Abstract. We discuss the dynamical behavior of two dimensional Hénon map. We studied four one-pieces attractors other than Hénon attractor and it's dynamical behavior. Also we studied the dynamical behavior of Hénon map for the different values of a for each fixed values of b as well as we find the range of parameter a that is $1 < a < 2$ with the given range $|b| < 1$.

Keyword: Hénon map, attractor, sink, saddle, saddle-node bifurcation, period-doubling bifurcation.

AMS Mathematics Subject Classification (2010): 37C25, 37C27, 37C70

1. Introduction

In 1963, the meteorologist Lorenz [03] published a numerical study of the ordinary differential equations

$$\frac{dx}{dt} = \sigma(y - x), \quad \frac{dy}{dt} = \rho x - y - xz, \quad \frac{dz}{dt} = -\beta z + xy.$$

where $\sigma, \rho, \beta > 0$ are parameters, obtained from the Oberbeck-Boussinesq equations for fluid convection in a two-dimensional layer heated from below and cooled from above. He used the parameters $\sigma = 10, \rho = 28$ and $\beta = 8/3$, and found a "strange attractor", later called the Lorenz attractor, where the solutions seemed to be attracted to a branched manifold and orbits were sensitive to initial conditions.

In 1976, Hénon [05] tried to understand the structure of the Lorenz attractor by a Poincaré map. As a model he used the polynomial map in the plane given by

$$(x, y) \mapsto (1 + y - ax^2, bx), \text{ where } a \text{ and } b \text{ are parameters.}$$

He found numerically a "strange attractor" with $a = 1.4$ and $b = 0.3$, later called the Hénon attractor. The mathematical nature of this attractor is still not known, but some progress has been made by Benedicks and Carleson [06].

Hénon observed that the attractor is very complicated dynamics. Many numerical studies were carried out in the late 1970s and early 1980s. The Hénon map has been the source of many numerical experiments and theoretical works.

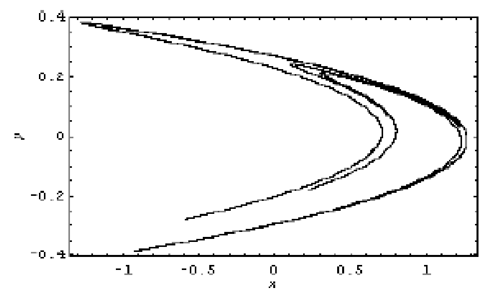

Fig. 1. Hénon attractor for $H_{a,b} = (1 + y - ax^2, bx)$ with $a = 1.4$ and $b = 0.3.$

We observed that there is no attractor other than Hénon attractor. So, we try to find another attractor of Hénon map and find some attractors of Hénon map with the parameter values $a = 1.07, b = 0.5;$ $a = 1.24, b = 0.4;$ $a = 1.61, b = 0.2;$ and $a = 1.80,$ $b = 0.1.$ Also we study the dynamical behavior of those attractors. We consider equivalent formulation of Hénon map

$H_{a,b} : \mathbf{R}^2 \to \mathbf{R}^2,$ $H_{a,b}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} a + by - x^2 \\ x \end{pmatrix},$ where a and b are real parameters.

2. Preliminaries

Definition 2.1. Let $f = (f_1, f_2, \cdots, f_m)$ be a map on $\mathbf{R}^m$, and let $p \in \mathbf{R}^m$. The Jacobian matrix of f at p, denoted $Df(p)$ is the matrix

$$Df(p) = \begin{pmatrix} \frac{\partial f_1}{\partial x_1}(p) \cdots \frac{\partial f_1}{\partial x_m}(p) \\ \vdots \qquad\quad \vdots \\ \frac{\partial f_m}{\partial x_1}(p) \cdots \frac{\partial f_m}{\partial x_m}(p) \end{pmatrix}$$ of partial derivatives evaluated at p.

Definition 2.2. Let f be a map on $\mathbf{R}^m, m \geq 1.$ Assume that $f(p) = p.$ Then the fixed point p is called hyperbolic if none of the eigenvalue of $Df(p)$ has magnitude 1. If p is hyperbolic and if at least one eigenvalue of $Df(p)$ has magnitude greater than 1 and at least one eigenvalue has magnitude less than 1, then p is called a saddle.

Theorem 2.3. Let f be a map on $\mathbf{R}^m$, and assume $f(p) = p$.
1. If the magnitude of each eigenvalue of $Df(p)$ is less than 1, then p is a sink or attracting.
2. If the magnitude of each eigenvalue of $Df(p)$ is greater than 1, then p is a source or repelling.

Proof: The proof of the theorem can be found in [01].

Definition 2.4. Given $x_0 \in R$, we define the orbit of x_0 under f to be the sequence of points $x_0,\ x_1 = f(x_0),\ x_2 = f^2(x_0), \cdots, x_n = f^n(x_0)$. The starting point x_0 is called the seed or initial value of the orbit.

Definition 2.5. Let f be a smooth map of the real line $\mathbf{R}$. The Lyapunov number $L(x_1)$ of the orbit $\{x_1, x_2, x_3, \cdots\}$ is defined as

$$L(x_1) = \lim_{n\to\infty}(|f'(x_1)| \cdots |f'(x_n)|)^{1/n},\ \text{if this limit exists.}$$

The Lyapunov exponent $h(x_1)$ is defined as

$$h(x_1) = \lim_{n\to\infty}\left(\frac{\ln|f'(x_1)| + \cdots + \ln|f'(x_n)|}{n}\right),\ \text{if this limit exists.}$$

Notice that h exists if and only if L exists, and $h = \ln L$.
It follows from the definition that the Lyapunov number of a fixed point x_1 for a one-dimensional map f is $|f'(x_1)|$, or equivalently, the Lyapunov exponent of the orbit is

$$h = \ln|f'(x_1)|.$$

Definition 2.6. Let f be a smooth map. An orbit $\{x_1, x_2, x_3, \cdots, x_n, \cdots\}$ is called asymptotically periodic if it converges to a periodic orbit as $n \to \infty$; this means that there exists a periodic orbit $\{y_1, y_2, y_3, \cdots, y_k, y_1, y_2, \cdots\}$ such that

$$\lim_{n\to\infty}|x_n - y_n| = 0.$$

Any orbit that is attracted to a sink is asymptotically periodic.

Definition 2.7. Let f be a map of $\mathbf{R}^m, m \ge 1$, and let $\{v_0, v_1, v_2, \cdots\}$ be a bounded orbit of f. The orbit is **chaotic** if

1. it is not asymptotically periodic,
2. no Lyapunov number is exactly one, and
3. the Lyapunov exponent $h(v_0)$ is greater than zero **(Lyapunov).**

Definition 2.8. Let $f: D \to D$ and $g: E \to E$ be functions. Then f is topologically conjugate to g if there is a homeomorphism $\tau: D \to E$ such that $\tau \circ f = g \circ \tau$. In this case, τ is called a topologically conjugacy.

3. Dynamical behaviors of the Hénon map

Theorem 3.1. For $|b|<1$, and $a_0(b) = -\frac{1}{4}(b-1)^2$

(a) if $a < a_0(b)$, $H_{a,b}$ has no fixed point,

(b) if $a = a_0(b)$, $H_{a,b}$ has exactly one fixed point,

(c) if $a > a_0(b)$, $H_{a,b}$ has exactly two fixed points,

Proof: Suppose $\begin{pmatrix} x \\ y \end{pmatrix} \in \mathbf{R}^2$ is a fixed point of $H_{a,b}$. Then $H_{a,b}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x \\ y \end{pmatrix}$

$$\Leftrightarrow x = a + by - x^2 \text{ and } y = x$$
$$\Leftrightarrow x^2 - (b-1)x - a = 0$$
$$\Leftrightarrow x \in \{x_+, x_-\} \text{ with } x_\pm = x_\pm(a,b) = \frac{(b-1) \pm \sqrt{(b-1)^2 + 4a}}{2}.$$

We obtain the equation $(b-1)^2 + 4a = 0$, which implies that

$$a = -\frac{(b-1)^2}{4} = a_0(b).$$

(a) For $a < a_0(b)$, $x_\pm(a,b) \notin \mathbf{R}^2$, so in this case there is no fixed point for

$$a < a_0(b).$$

(b) For $a = a_0(b)$, we obtain $x_+(a,b) = x_-(a,b) = \frac{(b-1)}{2}$.

In this case, we have exactly one fixed point $p_0(a,b) = \begin{pmatrix} \frac{(b-1)}{2} \\ \frac{(b-1)}{2} \end{pmatrix}$

(c) For $a > a_0(b)$, we have $x_+(a,b)$ and $x_-(a,b)$ in $\mathbf{R}^2$ and $x_+(a,b) \neq x_-(a,b)$.

So in this case, we have two fixed points of the form

$p_+(a,b) = \begin{pmatrix} x_+(a,b) \\ x_+(a,b) \end{pmatrix}$ and $p_-(a,b) = \begin{pmatrix} x_-(a,b) \\ x_-(a,b) \end{pmatrix}$, where $x_\pm(a,b)$ are as in above for $a > a_0(b)$.

Let $F \in C^1(\mathbf{R}^2, \mathbf{R}^2)$ be a bijective map and $x_0 \in \mathbf{R}^2$. Let us denote the eigenvalues of $DF(x_0)$ by $\lambda_i(F, x_0)$, $i = 1, 2$ and $F^n(x_0) = x_0$ $(n \in \mathbf{Z})$. Then by the Theorem 2.3, x_0 is

(i) Attracting n-periodic point of F if $\forall i \in \{1,2\}, \left|\lambda_i(F^n, x_0)\right| < 1$.

(ii) Repelling n-periodic point of F if $\forall i \in \{1,2\}, \left|\lambda_i(F^n, x_0)\right| > 1$.

(iii) Saddle n-periodic point of F if $\exists\ i \in \{1,2\}$ such that $\left|\lambda_i(F^n, x_0)\right| < 1$ and $\left|\lambda_i(F^n, x_0)\right| > 1$.

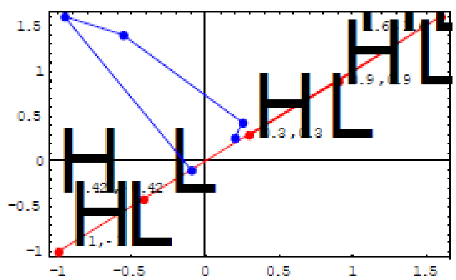

Fig. 2. Fixed points and period-2 points for $H_{a,b}$ with $b = 0.4$ and $a = -0.09,\ 0.27, 0.43,\ 1.6.$

Points along red line are fixed points and also points along blue line are period-2 points.

3.2. Chaotic behavior of the Hénon map

In the following, we use f instead of H for our convenience.

Let $f_{a,b}(x, y) = (a + by - x^2, x)$, where a and b are parameter.

(i) For the parameter values $a = 1.4$, $b = 0.3$, $f_{a,b}$ has the two fixed points $(-2, -2)$ and $(0.7, 0.7)$. The Jacobian matrix Df is $Df(x, y) = \begin{pmatrix} -2x & b \\ 1 & 0 \end{pmatrix}$. Evaluated at $(0.7, 0.7)$ the Jacobian matrix Df is $Df(0.7, 0.7) = \begin{pmatrix} -1.4 & 0.3 \\ 1 & 0 \end{pmatrix}$, with eigenvalues approximately equal to -1.59, and 0.19. [01]

The Lyapunov numbers of Hénon map are $L_1 = 1.59$ and $L_2 = 0.19$. The Lyapunov exponents of Hénon map are $h_1 = \ln 1.59 \approx 0.46$ and $h_2 = \ln 0.19 \approx -1.66$. Since every orbit of the Hénon map has a positive Lyapunov exponent and every orbit that is not asymptotically periodic, it will be chaotic.

(ii) For the parameter values $a = 1.24,\ b = 0.4,\ f_{a,b}$ has the two fixed points $(-1.45, -1.45)$ and $(0.85, 0.85)$. The Jacobian matrix Df is $Df(x, y) = \begin{pmatrix} -2x & b \\ 1 & 0 \end{pmatrix}$. Evaluated at $(0.85, 0.85)$ the Jacobian matrix Df is $Df(0.7, 0.7) = \begin{pmatrix} -1.7 & 0.4 \\ 1 & 0 \end{pmatrix}$, with eigenvalues approximately equal to -1.91, and 0.21. The Lyapunov numbers of Hénon map are $L_1 = 1.91$ and $L_2 = 0.21$. The Lyapunov exponents of Hénon map are $h_1 = \ln 1.91 \approx 0.64$ and $h_2 = \ln 0.21 \approx -1.56$. Since every orbit of the Hénon map has a positive Lyapunov exponent and every orbit that is not asymptotically periodic, it will be chaotic. Therefore, the Hénon map is chaotic on the square whose side length is 2.

4. Attractors of the Hénon map for the parameters values $-1 < a < 2,\ |b| < 1$.

4.1. For the parameter value $b = 0.7$, we can show the following results for the different values of parameter a from the calculations made in the following table by using Mathematica.

(i) If the parameters value $a = 0.18,\ b = 0.7$, the Hénon map $f_{a,b}$ will have only saddle fixed points and the period-two orbits are sink.

(ii) If the parameters value $a = 0.95,\ b = 0.7$, the Hénon map $f_{a,b}$ will have only saddle fixed points and the period-two orbits are saddle.

The value of a and $b = 0.7$	Fixed points	Eigenvalues	Nature of fixed points	Nature of period two orbits
0.18	$(-0.6, -0.6)$	$\{1.63, -0.43\}$	Saddle	Sink
	$(0.3, 0.3)$	$\{-1.19, 0.59\}$	Saddle	
0.95	$(-1.14, -1.14)$	$\{2.55, -0.27\}$	Saddle	Saddle
	$(0.84, 0.84)$	$\{-2.03, 0.35\}$	Saddle	

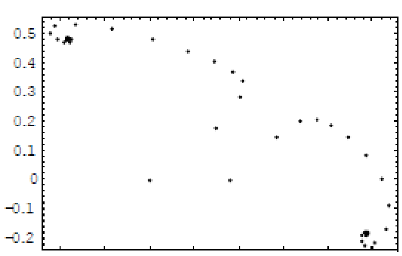

Fig. 3. Attractor for $a = 0.18,\ b = 0.7$

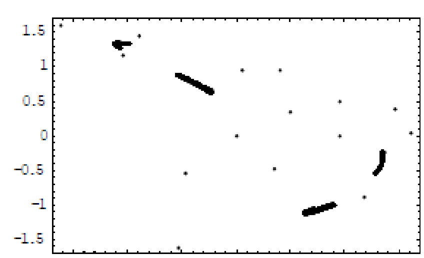

Fig. 4. Attractor for $a = 0.95,\ b = 0.7$

Above Fig. 3 have period-two orbits which are sink and the Fig. 4 has four pieces attractor which has saddle fixed points and period-two orbits are also saddle.

4.2. For the parameter value $b = 0.6$, we can show the following results for the different values of parameter a from the calculations made in the following table by using Mathematica.

(i) If the parameter values $0.04 < a < 0.12,\ b = 0.6,$ the Hénon map $f_{a,b}$ will have one sink fixed point and one saddle fixed point.

(ii) If the parameters value $a = -0.04,\ b = 0.6,$ the Hénon map $f_{a,b}$ will occur a saddle-node bifurcation.

(iii) If the parameters value $a = 0.12,\ b = 0.6,$ a period-doubling bifurcation occurs when the fixed point loses stability and a period-two orbit is born.

(iv) If the parameter values $a > 0.12,\ b = 0.6,$ the Hénon map $f_{a,b}$ will have saddle fixed points and period-two orbits are saddle.

The value of a and $b = 0.6$	Fixed points	Eigenvalues	Nature of fixed points	Nature of period two orbit	Bifurcation analysis
-0.04	$(-0.2, -0.2)$	$\{1, -0.6\}$			Saddle-node
0.50	$(-0.5, -0.5)$	$\{1.42, -0.42\}$	Saddle		
	$(0.1, 0.1)$	$\{-0.88, 0.68\}$	Sink		
0.12	$(-0.6, -0.6)$	$\{1.58, 0.38\}$	Saddle		
	$(0.2, 0.2)$	$\{-1, 0.6\}$			Period-doubling
0.13	$(-0.61, -0.61)$	$\{1.60, 0.38\}$	Saddle	Saddle	
	$(0.21, 0.21)$	$\{-1.01, 0.59\}$	Saddle		

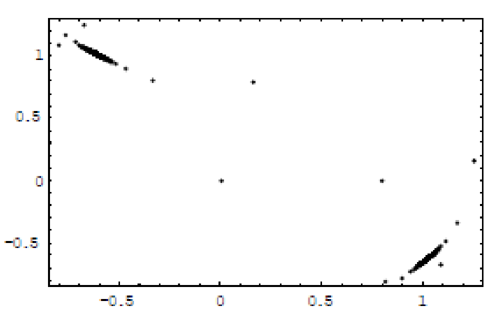

Fig. 5. Attractor for $a = 0.80,\ b = 0.6.$

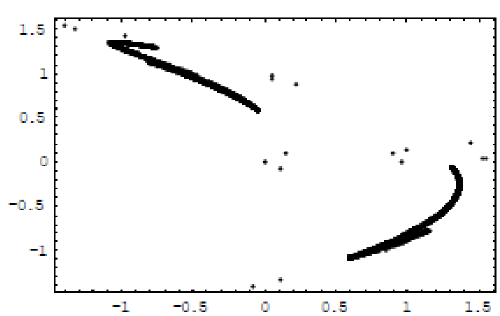

Fig. 6. Attractor for $a = 0.95,\ b = 0.6.$

Above Fig. 5 has a period-two orbit which sink. In the Fig. 6, there is two-pieces attractor which has saddle fixed points and period-two orbits are also saddle.

4.3. For the parameter value $b = 0.5$, we can show the following results for the different values of parameter a from the calculations made in the following table by using Mathematica.

(i) If the parameter values $0.06 < a < 0.19$, $b = 0.5$, the Hénon map $f_{a,b}$ has one sink fixed point and one saddle fixed point.

(ii) If the parameters value $a = -0.0625$, $b = 0.5$, a saddle-node bifurcation will occur.

(iii) If the parameters value $a = 0.19$, $b = 0.5$, a period-doubling bifurcation occurs when the fixed point loses stability and a period-two orbit is born.

(iv) If the parameter values $0.19 < a < 0.59$, $b = 0.5$, the Hénon map $f_{a,b}$ will have saddle fixed points and a period-two orbit is sink.

(v) If the parameter values $0.98 < a < 1.08$, $b = 0.5$, the Hénon map $f_{a,b}$ will have saddle fixed points and a period-two orbit is also a saddle.

The value of a and $b = 0.5$	Fixed points	Eigenvalues	Nature of fixed points	Nature of period two orbit	Bifurcation analysis
-0.0625	$(-0.25, -0.25)$	$\{1, -0.5\}$			Saddle-node
0.07	$(-0.61, -0.61)$	$\{1.54, -0.32\}$	Saddle		
	$(0.11, 0.11)$	$\{-0.83, 0.61\}$	Sink		
0.19	$(0.25, 0.25)$	$\{-1, 0.5\}$			Period-doubling
0.20	$(-0.76, -0.76)$	$\{1.80, -0.27\}$	Saddle	Sink	
	$(0.26, 0.26)$	$\{-1.013, 0.49\}$	Saddle		
0.99	$(-1.27, -1.27)$	$\{2.72, -0.183\}$	Saddle		
	$(0.77, 0.77)$	$\{-1.82, 0.28\}$	Saddle	Saddle	

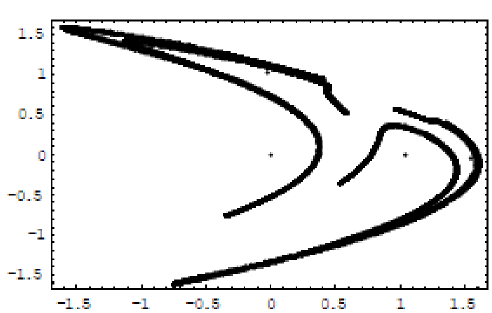

Fig. 7. Attractor for $a = 1.03$, $b = 0.5$

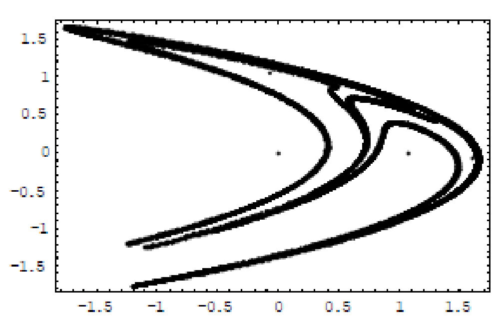

Fig. 8. Attractor for $a = 1.07$, $b = 0.5$

Above Fig. 7 has two-pieces attractor for $a = 1.03,\ b = 0.5.$ The points of an orbit alternate between the pieces. In the Fig. 8, two-pieces have merged to form one-piece attractor for $a = 1.07,\ b = 0.5.$

4.4. For the parameter value $b = 0.4,$ we can show the following results for the different values of parameter a from the calculations made in the following table by using Mathematica.

(i) If the parameter values $0.09 < a < 0.27,\ b = 0.4,$ the Hénon map $f_{a,b}$ will have one sink fixed point and one saddle fixed point.

(ii) If the parameters value $a = -0.09,\ b = 0.4,$ a saddle-node bifurcation will occur.

(iii) If the parameters value $a = 0.27,\ b = 0.4,$ a period-doubling bifurcation occurs when the fixed point loses stability and a period-two orbit is born.

(iv) If the parameter values $0.27 < a < 0.85,\ b = 0.4,$ the Hénon map $f_{a,b}$ will have saddle fixed points and period-two orbits are sink.

(v) If the parameter values $0.85 < a < 1.25,\ b = 0.4,$ the Hénon map $f_{a,b}$ will have saddle fixed points and period-two orbits are also saddle.

The value of a and $b = 0.4$	Fixed points	Eigenvalues	Nature of fixed points	Nature of period-two orbits	Bifurcation analysis
0.10	$(-0.74, -0.74)$	$\{1.71, -0.23\}$	Saddle		
-0.09	$(0.14, 0.14)$	$\{-0.79, 0.51\}$	Sink		
	$(-0.3, -0.3)$	$\{1, -0.4\}$			Saddle-node
0.27	$(0.3, 0.3)$	$\{-1, 0.4\}$			Period-doubling
0.28	$(-0.91, -0.91)$	$\{2.018, -0.20\}$	Saddle	Sink	
	$(0.31, 0.31)$	$\{-1.014, 0.39\}$	Saddle		
0.86	$(-1.27, -1.27)$	$\{2.69, -0.15\}$	Saddle	Saddle	
	$(0.67, 0.67)$	$\{-1.59, 0.25\}$	Saddle		

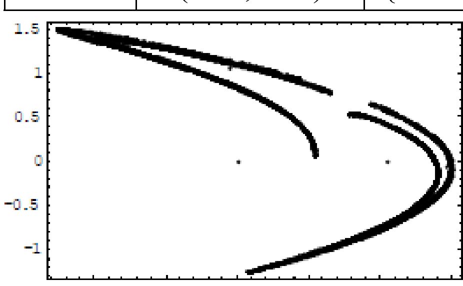

Fig. 9. Attractor for $a = 1.05,\ b = 0.4$

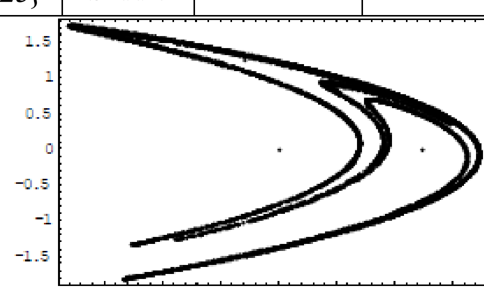

Fig. 10. Attractor for $a = 1.24,\ b = 0.4$

Above Fig. 9 has two-pieces attractor for $a = 1.045,\ b = 0.4.$ The points of an orbit alternate between the pieces. In the Fig. 10, two-pieces have merged to form one-piece attractor for $a = 1.24,\ b = 0.4.$

4.5. For the parameter value $b = 0.3,$ we can show the following results for the different values of parameter a from the calculations made in the following table by using Mathematica.

(i) If the parameter values $0.12 < a < 0.37,\ b = 0.3,$ the Hénon map $f_{a,b}$ will have one sink fixed point and one saddle fixed point.

(ii) If the parameters value $a = 0.37,\ b = 0.3$ a period-doubling bifurcation occurs when the fixed point loses stability and a period-two orbit is born.

(iii) If the parameter values $0.37 < a < 0.92,\ b = 0.3,$ the Hénon map $f_{a,b}$ will have saddle fixed points and a period-two orbit is sink.

(iv) If the parameter values $0.91 < a < 1.41,\ b = 0.3,$ the Hénon map $f_{a,b}$ will have saddle fixed points and period-two orbits are also saddle.

The value of a and $b = 0.3$	Fixed points	Eigenvalues	Nature of fixed points	Nature of period-two orbits	Bifurcation analysis
0.13	$(-0.85, -0.85)$	$\{1.86, -0.16\}$	Saddle		
	$(0.15, 0.15)$	$\{-0.72, 0.42\}$	Sink		
0.37	$(0.35, 0.35)$	$\{-1, 0.3\}$			Period-doubling
0.91	$(-1.37, -1.37)$	$\{2.85, -0.11\}$	Saddle	Sink	
	$(0.66, 0.66)$	$\{-1.52, 0.20\}$	Saddle		
1.40	$(-1.58, -1.58)$	$\{3.25, -0.09\}$	Saddle	Saddle	
	$(0.88, 0.88)$	$\{-1.91, 0.15\}$	Saddle		

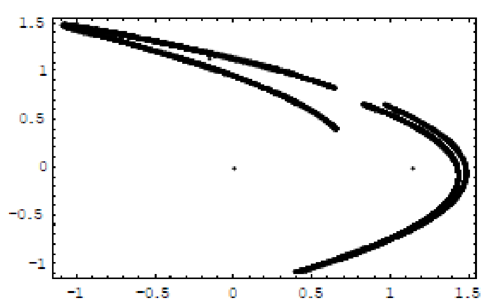

Fig. 11. Attractor for $a = 1.14,\ b = 0.3$

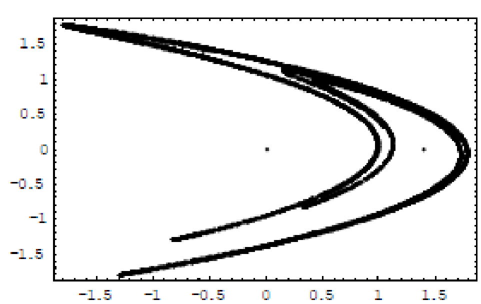

Fig. 12. Attractor for $a = 1.4,\ b = 0.3$

Above Fig. 11 has two-pieces attractor for $a = 1.14,\ b = 0.3.$ The points of an orbit alternate between the pieces. In the Fig. 12, two-pieces have merged to form one-piece attractor for $a = 1.4,\ b = 0.3.$

4.6. For the parameter value $b = 0.2,$ we can show the following results for the different values of parameter a from the calculations made in the following table by using Mathematica.

(i) If the parameter values $0.16 < a < 0.48,\ b = 0.2,$ the Hénon map $f_{a,b}$ will have one sink fixed point and one saddle fixed point.

(ii) If the parameters value $a = -0.16,\ b = 0.2,$ a saddle-node bifurcation will occur.

(iii) If the parameters value $a = 0.48,\ b = 0.2,$ a period-doubling bifurcation occurs when the fixed point loses stability and a period-two orbit is born.

(iv) If the parameter values $0.48 < a < 1.01,\ b = 0.2,$ the Hénon map $f_{a,b}$ will have a saddle fixed point and a period-two orbit sinks.

(v) If the parameter values $1.00 < a < 1.62,\ b = 0.2,$ the Hénon map $f_{a,b}$ will have saddle fixed points and a period-two orbit is also a saddle.

The value of a and $b = 0.2$	Fixed points	Eigenvalues	Nature of fixed points	Nature of period-two orbits	Bifurcation analysis
-0.16	$(-0.4, -0.4)$	$\{1, -0.2\}$			Saddle-node
0.17	$(-0.97, -0.97)$	$\{2.04, -0.09\}$	Saddle		
	$(0.17, 0.17)$	$\{-0.65, 0.31\}$	Sink		
0.48	$(0.4, 0.4)$	$\{-1, 0.2\}$			Period-doubling
0.99	$(-1.47, -1.47)$	$\{3.01, -0.07\}$	Saddle	Sink	
	$(0.67, 0.67)$	$\{-1.47, 0.14\}$	Saddle		
1.61	$(-1.73, -1.73)$	$\{3.52, -0.06\}$	Saddle	Saddle	
	$(0.93, 0.93)$	$\{-1.96, 0.10\}$	Saddle		

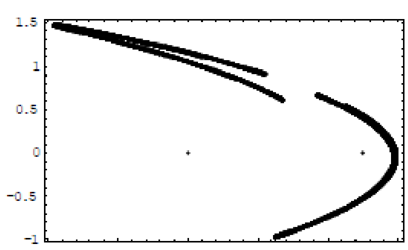

Fig. 13. Attractor for $a = 1.26,\ b = 0.2$

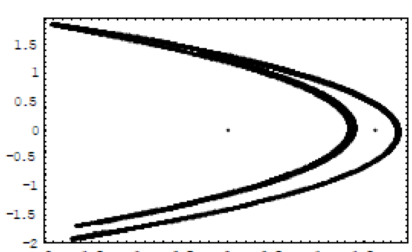

Fig. 14. Attractor for $a = 1.61,\ b = 0.2$

Above Fig. 13 has two-pieces attractor for $a=1.26,\ b=0.2.$ The points of an orbit alternate between the pieces. In the Fig. 14, two-pieces have merged to form one-piece attractor for $a=1.61,\ b=0.2.$

4.7. For the parameter value $b=0.1,$ we can show the following results for the different values of parameter a from the calculations made in the following table by using Mathematica.

(i) If the parameter values $0.20<a<0.61,\ b=0.1,$ the Hénon map $f_{a,b}$ will have one sink fixed point and one saddle fixed point.

(ii) If the parameters value $a=0.61,\ b=0.1,$ a period-doubling bifurcation occurs when the fixed point loses stability and a period-two orbit is born.

(iii) If the parameter values $0.61<a<1.44,\ b=0.1,$ the Hénon map $f_{a,b}$ will have saddle fixed points and a period-two orbit is sink.

(iv) If the parameter values $1.13<a<1.81,\ b=0.1,$ the Hénon map $f_{a,b}$ will have saddle fixed points and a period-two orbit is also a saddle.

The value of a and $b=0.1$	Fixed points	Eigenvalues	Nature of fixed points	Nature of period-two orbits	Bifurcation analysis
0.60	$(-1.34,-1.34)$	$\{2.72,-0.04\}$	Saddle		
	$(0.44,0.44)$	$\{-0.98,0.10\}$	Sink		
0.61	$(0.45,0.45)$	$\{-1,0.1\}$			Period-doubling
1.12	$(-1.47,-1.47)$	$\{3.01,-0.07\}$	Saddle	Sink	
	$(0.7,0.7)$	$\{-1.47,0.07\}$	Saddle		
1.80	$(-1.86,-1.86)$	$\{3.74,-0.03\}$	Saddle	Saddle	
	$(0.96,0.96)$	$\{-1.97,0.05\}$	Saddle		

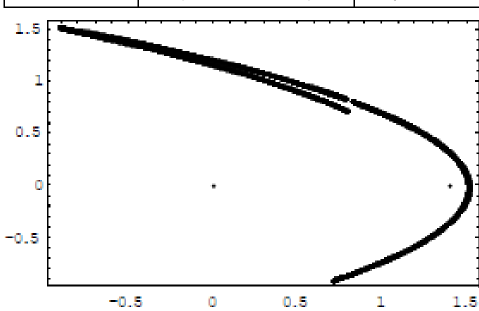

Fig. 15. Attractor for $a=1.40,\ b=0.1$

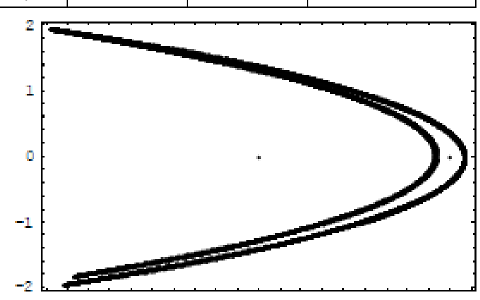

Fig. 16. Attractor for $a=1.80,\ b=0.1$

Above Fig. 15 has two-pieces attractor for $a = 1.40,\ b = 0.1.$ The points of an orbit alternate between the pieces. In the Fig. 16, two-pieces have merged to form one-piece attractor for $a = 1.80,\ b = 0.1.$

5. Conclusion

From the above study we can conclude our result as follows:

(i) The Henon map $f_{a,b}$ has attractors with the parameter values $-1 < a < 2$ and $|b| < 1.$ In those attractor, four one-pieces attractors are found with the parameter values $1 < a < 2$ and $|b| < 1.$

(ii) The Hénon map $f_{a,b}$ has two fixed points and period-two orbits which are called saddle with the parameter values $0.85 < a < 2$ and $|b| < 1.$

(iii) For the parameters values $a = 0.12, b = 0.6;$ $a = 0.19, b = 0.5;$ $a = 0.27, b = 0.4;$ $a = 0.37,\ b = 0.3;$ $a = 0.48, b = 0.2;$ and $a = 0.61, b = 0.1,$ a period-doubling bifurcation occurs when the fixed point loses stability and a period-two orbit is born.

(iv) The Hénon map is chaotic on the square whose side length 2 for the parameter values $a > 0$ and $|b| < 1.$

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