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Real Neutrosophic Matrix and Real Nilpotent Neutrosophic Matrix

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Abstracts. Real neutrosophic nilpotent matrices are a special class of matrices characterized by their neutrosophic components, which quantify the indeterminacy in their elements. This paper explores the properties and important results related to real neutrosophic nilpotent matrices. Specifically, it investigates their structure, determinacy indices, and relationships under matrix multiplication. The rank, index, and linear transformations associated with these matrices are analyzed in the context of neutrosophic logic, highlighting their implications in mathematical modeling and decision-making under uncertainty. Furthermore, the study extends to Green's relations and semigroups, elucidating how neutrosophic nilpotent matrices contribute to the understanding of algebraic structures with indeterminate elements. The findings provide a comprehensive framework for applying neutrosophic logic in matrix theory and its broader applications in diverse fields.

Keywords: Real nilpotent netrosophic matrix; rank; index; linear transformation; Green's relation; semigroup

AMS Mathematics Subject Classification (2010): 94D05

Abbreviation	Meaning
NSs	Neutrosophic set
SVNS	Single valued neutrosophic set
NNs	Neutrosophic number
NFM	Neutrosophic fuzzy matrix
RNM	Real neutrosophic matrix

1. Introduction

After the establishment of fuzzy set (FS) theory, it became clear that while FS effectively addressed many problems involving non-random uncertainty, it had limitations in handling cases where information was incomplete or inconsistent. In 1983, Atanassov introduced intuitionistic fuzzy sets (IFS) as an extension of FS, incorporating membership and non-membership values that sum up to less than or equal to 1. When these values sum exactly to 1 for all members, IFS reduces back to FS. This extension aimed to address situations where FS fell short due to inadequate information.

However, IFS itself proved insufficient in scenarios involving more complex uncertainties. To tackle these challenges, Smarandache introduced neutrosophic sets (NS) in the late 1990s. NS are characterized by three parameters: truth membership function (t), indeterminacy membership function (i), and falsity membership function (f), which are subsets of the non-standard unit interval. This framework goes beyond IFS by accommodating uncertainties that extend beyond what IFS can manage effectively.

Neutrosophic sets generalize classical fuzzy sets, interval-valued fuzzy sets, and intuitionistic fuzzy sets, among others. They have been extended into various branches, such as interval neutrosophic sets and generalized neutrosophic soft sets, demonstrating their versatility in handling diverse real-world scenarios involving indeterminate and inconsistent information.

The concept of single-valued neutrosophic sets (SVNS) was introduced to provide a more flexible representation where truth, indeterminacy, and falsity can be independently quantified for each element. This framework is particularly suited for modeling human reasoning processes, accommodating imperfect knowledge and uncertainties inherent in decision-making.

In recent years, there has been significant research interest in neutrosophic sets and their applications, leading to developments like neutrosophic fuzzy numbers (NFN) and corresponding matrices. These extensions build upon the foundations laid by FS and IFS, offering robust tools to address complex decision-making scenarios across various domains.

For further details on neutrosophic sets, numbers, and matrices, interested readers can refer to comprehensive resources on the subject.

2. Literature review

The concept of neutrosophic logic, introduced by [36], has broadened the scope of mathematical and logical frameworks by incorporating degrees of truth, falsity, and indeterminacy. This extension facilitates the handling of uncertainty and incomplete information, which classical fuzzy logic often struggles with. In this context, the notion of a real nilpotent neutrosophic matrix represents an intriguing convergence of matrix theory and neutrosophic logic, merging the properties of nilpotent matrices with the capabilities of neutrosophic structures.

A real nilpotent neutrosophic matrix is a specialized matrix that not only adheres to the characteristics of nilpotent matricesâ€”having a power that equals the zero matrixâ€”but also operates within the framework of neutrosophic logic. This combination introduces unique properties and challenges, making it a significant topic in contemporary mathematical research.

Neutrosophic logic expands on classical Boolean logic by including three components: truth membership, falsity membership, and indeterminacy membership. A neutrosophic set A is characterized by these three functionsâ€" $T_A(x)$, $F_A(x)$, and $I_A(x)$â€"where $T_A(x)$ denotes the degree of truth, $F_A(x)$ denotes the degree of falsity, and $I_A(x)$ represents the degree of indeterminacy. The sum $T_A(x) + F_A(x) + I_A(x)$ does not necessarily equal 1, which differentiates it from traditional fuzzy sets where this sum is constrained to equal 1[36] .

Nilpotent matrices, on the other hand, are matrices for which there exists a positive integer k such that the matrix raised to the power k results in the zero matrix. These matrices are characterized by their eigenvalues being zero and their Jordan canonical form consisting of Jordan blocks with zeros on the diagonal [34] . The concept of nilpotency plays a crucial role in various areas of linear algebra, including matrix theory and control theory.

The real nilpotent neutrosophic matrix merges these two concepts. For such a matrix, not only must the matrix be nilpotent (i.e., $A^k = 0$ for some k), but it must also be expressed in terms of neutrosophic numbers. These neutrosophic numbers have components that are not simply real numbers but include degrees of truth, falsity, and indeterminacy. This fusion of properties allows for a more nuanced representation of uncertainty and transformation properties in matrix theory.

Recent research has delved into characterizing the properties of real nilpotent neutrosophic matrices.[38] provided significant insights into the spectral properties of these matrices. They explored how the spectral characteristics of nilpotent matrices are preserved within the neutrosophic framework. Their work established criteria for nilpotency within the neutrosophic context and introduced algorithms for verifying nilpotency, expanding the practical applications of these matrices in handling uncertain and imprecise data.

The computational aspects of real nilpotent neutrosophic matrices are also a focal point of recent studies. The development of algorithms for matrix decomposition and similarity transformations has been crucial for practical applications.[35] introduced an efficient algorithm for computing the Jordan canonical form of real nilpotent neutrosophic matrices. This advancement addresses the complexity inherent in working with these matrices and provides a more streamlined approach for computations involving real nilpotent neutrosophic matrices. In practical applications, real nilpotent neutrosophic matrices have shown promise in various domains, including decision-making and control systems. In decision-making, these matrices can model complex systems with inherent uncertainties.[39] applied real nilpotent neutrosophic matrices to multi-criteria decision-making problems, demonstrating their effectiveness in scenarios characterized by vagueness and incomplete information. Their research illustrated how these matrices can be used to optimize decisions in environments where traditional methods may fall short.

In the realm of control systems, real nilpotent neutrosophic matrices offer a framework for designing controllers when faced with incomplete or uncertain information. [40] explored how these matrices can be utilized in control system design, particularly in situations where conventional linear matrix methods are inadequate. Their findings indicate that real nilpotent neutrosophic matrices can enhance the robustness and flexibility of control systems in the presence of uncertainty.

Despite these advancements, there are still challenges associated with real nilpotent neutrosophic matrices. The primary challenge lies in the complexity of computations involving these matrices, particularly as the dimensions of the matrices increase. Future research is likely to focus on developing more efficient algorithms and expanding the theoretical understanding of these matrices to address such challenges. Additionally, exploring new applications in emerging fields such as artificial intelligence and big data analytics represents a promising avenue for future work.

3. Neutrosophic number

Samrandche first proposed a concept of **neutrosophic number** which consists of the determinant part and the indeterminate part. It is usually denoted by $N = a + bI$, where a and b are real numbers and I is the indeterminacy such that $I^2 = I, I.0 = 0$ and $\frac{I}{I}$ is undefined. We call $N = a + bI$ as a pure neutrosophic number if $a = 0$.

For example, we consider a neutrosophic number $N = 5 + 3I$. If $I \in [0,0.02]$, then it is equivalent to $N \in [5,5.06]$ for $N \geq 5$. This means the determinant part is 5, whereas the indeterminacy part is $3I$ for $I \in [0,0.02]$, which means the possibility for number N to be a little bigger than 5.

Note that this number looks like a complex number, but, see that here $I^2 = I$, not -1 like a complex number.

The three basic operators defined on neutrosophic numbers $P = p_1 + q_1 I$ and $Q = p_2 + q_2 I$ are as follows:

(i) $P + Q = (p_1 + p_2) + (q_1 + q_2)I$
(ii) $P - Q = (p_1 - p_2) + (q_1 - q_2)I$
(iii) $P \times Q = p_1 p_2 + (p_1 q_2 + q_1 p_2 + q_1 q_2)I$

In real neutrosophic algebra, we denote K as the neutrosophic field over some neutrosophic vector spaces. We call the smallest field generated by $K \cup I$ or $K(I)$ to be the neutrosophic field for it involves the indeterminacy factor in it, where I has the special property that $I^n = I, I + I = I$ and if $t \in K$ be some scalar then $t.I = tI, 0.I = 0$. Thus, we generally denote neutrosophic field $K(I)$ generated by $K \cup I$, i.e. $K(I) = \langle K \cup I \rangle$.

Thus, for different fields of algebra, we can define several types of neutrosophic field generated by the field of neutrosophic vector space.

1. **R** be the field of real numbers, then the neutrosophic field generated by $\langle R \cup I \rangle$ is $R(I)$ and $R \subset R(I)$.
2. **Q** be the field of rational number, then the neutrosophic field generated by $\langle Q \cup I \rangle$ is $Q(I)$ and $Q \subset Q(I)$.
3. **Z** be the field of integers, then the neutrosophic field generated by $\langle Z \cup I \rangle$ is $\mathbf{Z}(I)$.

Thus, we can formulate the following set of neutrosophic numbers as given below:

$$\mathbf{R}(I) = \{a + bI: a, b \in \mathbf{R}\}$$
$$\mathbf{Q}(I) = \{a + bI: a, b \in \mathbf{Q}\}$$
$$\mathbf{Z}(I) = \{a + bI: a, b \in \mathbf{Z}\}$$
$$\mathbf{I}(I) = \{a + bI: a, b \in [0,1]\}.$$

So several types of neutrosophic numbers are available in the literature. However, many authors are confused about this classification. In this chapter, we will discuss first $\mathbf{I}(I)$, the fuzzy neutrosophic numbers (referred to as FNNs), and then we consider the matrix over real neutrosophic numbers (RNN) $\mathbf{R}(I)$.

3.1. Real neutrosophic matrix

Here we consider the neutrosophic matrix over real numbers based on the work of Smarandache [15]. So it is referred to as a real neutrosophic matrix and is abbreviated by RNM. For details of this matrix see [8].

The neutrosophic number over the field of real/complex numbers is defined in the form $a = a_1 + b_1 I$, where a_1, a_2 are real or complex numbers and I is the indeterminacy [5].

An RNM is defined as in FNM, i.e. of the form $M = M_1 + M_2 I$ where M_1 and M_2 are real matrices. The set of real matrices of order $m \times n$ is denoted by $\mathcal{M}_{mn}^R$ and that of order $n \times n$ by $\mathcal{M}_n^R$. The identity RNM of order $n \times n$ is denoted by U_n, all diagonal elements are 1 and all other elements are 0.

The null and identity matrices of order 3×3 are

$$O_3 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \ and \ \ U_3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

The basic operations on RNMs $M = M_1 + M_2 I$ and $N = N_1 + N_2 I$ are

(i) $M + N = (M_1 + N_1) + (M_2 + N_2)I$

(ii) $M - N = (M_1 - N_1) + (M_2 - N_2)I$

(iii) $MN = (M_1 N_1) + (M_2 N_1 + M_1 N_2 + M_2 N_2)I$. In this case also, $I^n = I^2 = I$ for any positive integer n.

Assumed that the order of RNMs is compatible with the appropriate operations.

This matrix looks like a complex matrix, but see that here I represents indeterminacy, not complex $i = \sqrt{-1}$. Also, $I^n = I$ for all positive integer n, which is not true for complex numbers.

Let $M = M_1 + M_2 I$ be a RNM, where $M_1, M_2 \in \mathcal{M}_n^R$. Then its determinant is denoted by $det(M)$ or $|M|$ and its value is given by

$$det(M) = det(M_1) + I[det(M_1 + M_2) - det(M_1)]. \tag{1}$$

Note that this formula is unlike to the determinant of conventional matrix. But, this definition follows the rules of conventional matrices.

Lemma 1. *Let $M = M_1 + M_2 I$ and $N = N_1 + N_2 I$ be two RNMs, where $M_1, M_2, N_1, N_2 \in \mathcal{M}_n^R$. Then $det(MN) = det(M)det(N)$.*

Proof: By definition of product

$$MN = M_1 N_1 + [M_1 N_2 + M_2 N_1 + M_2 N_2]I.$$

Now, by Definition (1) of determinant,

$$\begin{aligned} det(MN) &= det(M_1 N_1) + [det(M_1 N_1 + M_1 N_2 + M_2 N_1 + M_2 N_2) - det(M_1 N_1)]I \\ &= det(M_1 N_1) + [det(M_1 + M_2)(N_1 + N_2) - det(M_1 N_1)]I. \end{aligned}$$

Again,

$$\begin{aligned} det(M)det(N) &= det(M_1 N_1) + [det\{(M_1 + M_2)N_1\} - det(M_1 N_1) \\ &+ det\{M_1(N_1 + N_2)\} - det(M_1 N_1) + det\{(M_1 + M_2)(N_1 + N_2)\} \\ &- det\{M_1(N_1 + N_2)\} - det\{N_1(M_1 + M_2)\} + det(M_1 N_1)]I \\ &= det(M_1 N_1) + [det(M_1 + M_2)(N_1 + N_2) - det(M_1 N_1)]I \\ &= det(MN). \end{aligned}$$

The orthogonal RNM can also be defined like conventional matrices. Let $M = M_1 + M_2 I$, where $M_1, M_2 \in \mathcal{M}_n^R$ be an RNM. The RNM M is called orthogonal if $MM^T = U_n$.

If $MM^T = U_n$, then $MM^T = M_1 M_1^T + (M_1^T M_2 + M_1 M_2^T + M_2 M_2^T) I$. The necessary condition is that $M_1^T M_2 + M_1 M_2^T + M_2 M_2^T = 0$ and M_1 is orthogonal. Also, $det(MM^T) = det(U_n) = 1$. This implies $det(M_1) = 1$ and $det(M_1 + M_2) = 1$.

4. Real neutrosophic nilpotent matrix

Let A be a neutrosophic matrix, where

$$A = A_1 + A_2 I$$

then,

$$A^2 = (A_1 + A_2 I)(A_1 + A_2 I)$$
$$= A_1^2 + [A_1 A_2 + A_2 A_1 + A_2^2] I$$
$$= A_1^2 + [(A_1 + A_2)^2 - A_1^2] I$$

Again,

$$A^3 = A^2 \cdot A$$
$$= A_1^3 + [A_1^2 A_2 A_1 + A_2 A_1^2 + A_2^2 A_1 + A_1^2 A_2 + A_1 A_2^2 + A_2 A_1 A_2 + A_2^3] I$$
$$= A_1^3 + [(A_1 + A_2)^3 - A_1^3] I$$

In this way it can be shown that

$$A^n = A_1^n + [(A_1 + A_2)^n - A_1^n] I$$

This expression shows that if A_1 and $(A_1 + A_2)$ are nilpotent then A is nilpotent with same index.

Recall that a neutrosophic square matrix is nilpotent is some positive power of it is the zero matrix.

Let F be a field.then the following results is holds

(1) (a) Suppose that $(A_1 + A_2 I) \in F^n \times n$ has a nonzero eigenvalue $(\lambda_1 + \lambda_2 I)$. Find a vector $(x_1 + x_2 I)$ such that $(A_1 + A_2 I)^K (x_1 + x_2 I) \neq 0$ for all $k \in N$. Deduce that $(A_1 + A_2 I)$ is not nilpotent.

(b) Show that all neutrosophic eigenvalues of a nilpotent neutrosophic matrix are 0.

(c) Deduce, by proving the converse to (b), that a matrix $(A_1 + A_2 I) \in F$ is nilpotent neutrosophic if and only if all its eigenvalues are 0. (Hint: Cayley-Hamilton)

(d) Deduce that if $(A_1 + A_2 I) \in F$ satisfies $A_1 + A_2 I$=0 for some $m \in N$ then $(A_1 + A_2 I)^n$=0.

Proof:

(a)Take $(x_1 + x_2 I)$ to be an neutrosophic eigenvector of $(A_1 + A_2 I)$ with neutrosophic eigenvalue $(\lambda_1 + \lambda_2 I)$. Then $(A_1 + A_2 I)(x_1 + x_2 I) = (\lambda_1 + \lambda_2 I)(x_1 + x_2 I)$

or, $A_1 x_1 + [A_1 x_2 + A_2 x_1 + A_2 x_2] I = \lambda_1 x_1 + [\lambda_1 x_2 + \lambda_2 x_1 + \lambda_2 x_1] I$

This implies that ,$A_1 x_1 = \lambda_1 x_1$ and $A_1 x_2 + A_2 x_1 + A_2 x_2 = \lambda_1 x_2 + \lambda_2 x_1 + \lambda_2 x_1$

$$(A_1 + A_2)(x_1 + x_2) - A_1 x_1 = (\lambda_1 + \lambda_2)(x_1 + x_1) - \lambda_1 x_1$$

Since $A_1 x_1 = \lambda_1 x_1$ and $(A_1 + A_2)(x_1 + x_2) = (\lambda_1 + \lambda_2)(x_1 + x_2)$

and by induction, $A_1^k x_1 = \lambda_1^k x_1$ and $(A_1 + A_2)^k (x_1 + x_2) = (\lambda_1 + \lambda_2)^k (x_1 + x_2)$

Now, $(A_1 + A_2 I)^K (x_1 + x_2 I) = \{A_1^k + [(A_1 + A_2)^k - A_1^k] I\}(x_1 + x_2 I)$

$$= A_1^K x_1 + [(A_1 + A_2)(x_1 + x_2) - A_1^K x_1 + A_1^K x_2 - A_1^K x_2] I$$

$= A_1^K x_1 + [(A_1 + A_2)(x_1 + x_2) - A_1^K x_1]I$
$= \lambda_1^k x_1 + [(\lambda_1 + \lambda_2)^k (x_1 + x_2) - \lambda_1^k x_1]$
$= (\lambda_1 + \lambda_2 I)^k (x_1 + x_2 I)$

Therefore, $(A_1 + A_2 I)^K (x_1 + x_2 I) = (\lambda_1 + \lambda_2 I)^k (x_1 + x_2 I)$ for each $k \in N$. So no power of $(A_1 + A_2 I)$ can be the zero matrix. So $(A_1 + A_2 I)$ is not nilpotent.

(b) By (a), a nilpotent neutrosophic matrix can have no nonzero neutrosophic eigenvalues, i.e., all its neutrosophic eigenvalues are 0.

(c) Suppose $(A_1 + A_2 I)$has all neutrosophic eigenvalues equal to 0. Then the characteristic polynomial of $(A_1 + A_2 I)$ is $x(x) = [(x_1 + x_2 I) - (\lambda_{11} + \lambda_{21} I)] \ldots [(x_1 + x_2 I) - (\lambda_{1n} + \lambda_{2n} I)] = (x_1 + x_2 I)^n$ so, by the Cayley-Hamilton Theorem, $x(A_1 + A_2 I) = (A_1 + A_2 I)^n = 0$, making $(A_1 + A_2 I)$ nilpotent. Then, using (b), we have (c) in full.

(d) Follows from (c), as nilpotentcy implies$(A_1 + A_2 I)^n = 0$.
For a nilpotent neutrosophic matrix $(A_1 + A_2 I) \in F^n \times n$, denote by $v(A_1 + A_2 I)$ the least exponent m such that $(A_1 + A_2 I) = 0$. From Q1, we know that $v(A_1 + A_2 I) \leq n$. We investigate whether $(A_1 + A_2 I)$ can take all integer values 1,2... n.

(a) Describe the (i, j)th entry of $(A_1 + A_2 I)^m$ in terms of the entries of $(A_1 + A_2 I) = (a_{1ik_1} + a_{2ik_1} I)$.

(b) Use your description in (a) to show that the neutrosophic companion matrix-call it $(C_{1n} + C_{2n} I)$ of the polynomial $(x_1 + x_2 I)^n$ has $(C_{1n} + C_{2n} I)^{n-1} \neq 0$. Hence write down $v(C_{1n} + C_{2n} I) I)$. (Maybe try some small values of n first.)

(a)The (i,j)th entry of $(A_1 + A_2 I)^m$ is a sum of all possible nonzero products $(a_{1ik_1} + a_{2ik_1} I)(a_{1k_1k_2} + a_{2k_1K_2} I)(a_{1k_2k_3} + a_{2k_2k_3} I) \ldots\ldots (a_{1k_{(m-1)}j} + a_{2k_{(m-1)}j} I)$.

(b) The (1,n)th term of $(C_{1n} + C_{2n} I)^{n-1}$ is $(c_{112} + c_{212} I)(c_{123} + c_{223} I)(c_{134} + c_{234} I) \ldots\ldots (c_{1(n-1)n} + c_{2(n-1)_n} I) = 1 \cdot 1 \cdot 1 \cdot 1 = 1$ this being the only nonzero product, and so giving a nonzero term. Hence $v(C_{1n} + C_{2n} I) = n$.

(3) (a) For $(A_1 + A_2 I) \in \mathbb{C}^{n \times n}$ define $exp(A_1 + A_2 I)$, Show that if $(A_1 + A_2 I)$ is nilpotent neutrosophic then $exp(A_1 + A_2 I) = \sum_{i=0}^{n-1} (A_1 + A_2 I)^i / i!$

(b) For $(A_1 + A_2 I) \in \mathbb{C}^{n \times n}$ and nilpotent neutrosophic , show that

$$[U - (A_1 + A_2 I)]^{-1} = U + (A_1 + A_2 I) + (A_1 + A_2 I)^2 + \ldots\ldots + (A_1 + A_2 I)^{n-1}$$

where (as usual) U is thenxn neutrosophic identity matrix.

(a) We have $exp(A_1 + A_2 I)$ We have $exp(A_1 + A_2 I) = \sum_{i=0}^{\infty} (A_1 + A_2 I)^i / i!$, which reduces to $\sum_{i=0}^{n-1} (A_1 + A_2 I)^i / i!$ as $(A_1 + A_2 I)^n = 0$.

(b) We have,

$$[U - (A_1 + A_2 I)][U + (A_1 + A_2 I) + (A_1 + A_2 I)^2 + \ldots\ldots + (A_1 + A_2 I)^{n-1}]$$
$$= U + [(A_1 + A_2 I) - (A_1 + A_2 I)] + [(A_1 + A_2 I)^2 - (A_1 + A_2 I)^2] + \cdots + [(A_1 + A_2 I)^{n-1} - (A_1 + A_2 I)^{n-1}]$$
$$= U,$$

as$(A_1 + A_2 I)^n = 0$.This gives the result.

(4) For $(A_1 + A_2 I)$ and $(B_1 + B_2 I) in F^{n \times n}$ with $(A_1 + A_2 I)$ nilpotent and $(B_1 + B_2 I)$ nonsingular, show in two different ways that $(B_1 + B_2 I)^{-1}(A_1 + A_2 I)(B_1 +$

$B_2I)$ is nilpotent.

(i) By direct multiplication;

(ii) By considering the eigenvalues of $(B_1+B_2I)^{-1}(A_1+A_2I)(B_1+B_2I)$.

We have

$$\{(B_1+B_2I)^{-1}(A_1+A_2I)(B_1+B_2I)\}\cdot\{(B_1+B_2I)^{-1}(A_1+A_2I)(B_1+B_2I)\}\cdots\{(B_1+B_2I)^{-1}(A_1+A_2I)(B_1+B_2I)\}$$
$$=(B_1+B_2I)^{-1}(A_1+A_2I)^n(B_1+B_2I)$$
$$=(B_1+B_2I)^{-1}0(B_1+B_2I)=0.$$

But also, we know from lectures that $(B_1+B_2I)^{-1}(A_1+A_2I)(B_1+B_2I)$ and (A_1+A_2I) have the same eigenvalues, so by Q1, the neutrosophic eigenvalues of (A_1+A_2I) are all 0, and thus so are all the neutrosophic eigenvalues of $(B_1+B_2I)^{-1}(A_1+A_2I)(B_1+B_2I)$. Hence (again by Q1) $(B_1+B_2I)^{-1}(A_1+A_2I)(B_1+B_2I)$ is nilpotent.

5. Product of two real neutrosophic nilpotent matrix

5.1. Neutrosophic nilpotent transformation

In the notation established in [29], V denotes a vector space of dimension n over an arbitrary field, and $L(V)$ represents the semigroup of linear transformations from V to itself. Elements of $L(V)$ are denoted by symbols like $(\alpha_1+\alpha_2I)$,$(\beta_1+\beta_2I)$, $(\gamma_1+\gamma_2I)$, etc., and are written on the right of the argument to maintain consistency with transformation semigroup theory. If $(\alpha_1+\alpha_2I)\in L(V)$, then $ker(\alpha_1+\alpha_2I)$ and ran $(\alpha_1+\alpha_2I$)denote the kernel and range (or image) of $\alpha_1+\alpha_2I$, respectively;and we write $n(\alpha_1+\alpha_2I)=dimker(\alpha_1+\alpha_2I)$ for the nullity of $\alpha_1+\alpha_2I$ and $r(\alpha_1+\alpha_2I)=dimran(\alpha_1+\alpha_2I)$, for the rank of $\alpha_1+\alpha_2I$. If $(\alpha_1+\alpha_2I)\in$ L(V) and there exists $m\geq 1$ such that $(\alpha_1+\alpha_2I)^m=0$ but $(\alpha_1+\alpha_2I)^{m-1}\neq 0$, we say $(\alpha_1+\alpha_2I)$ is nilpotent with index m, and we denote by N(V) the subsemigroup of $L(V)$ generated by all nilpotent elements of $L(V)$.

Following [29], we adopt the convention from [22], where $\{e_{1i}+e_{2i}I\}$ stands for $\{(e_{1i}+e_{2i}I){:}i\in U\}$, with the subscript i representing an index set U. The subspace of V generated by a linearly independent subset $\{(e_{1i}+e_{2i}I)\}$ of V is denoted by $\langle e_i\rangle$. Sometimes, it is necessary to construct a transformation $(\alpha_1+\alpha_2I)\in L(V)$ by first choosing a basis $\{e_i\}$ for V and some elements $\{u_i\}$ V, and then defining $(\alpha_1+\alpha_2I)(e_{1i}+e_{2i}I)=(u_{1i}+u_{2i}I)$ for each $i\in U$ and extending this action by linearity to the whole of V. To simplify, within context, we say $(\alpha_1+\alpha_2I)\in L(V)$ is defined by letting

$$(\alpha_1+\alpha_2I)=\begin{pmatrix}(e_{1i}+e_{2i}I)\\(u_{1i}+u_{2i}I)\end{pmatrix}\tag{2}$$

given $\{(e_{1i}+e_{2i}I)\}$ and $\{(u_{1i}+u_{2i}I)\}$.

Green's relations are crucial in semigroup theory. Specifically, Green's $\mathcal{J}$ relation on L(V) satisfies the property: $(\alpha_1+\alpha_2I)$ $\mathcal{J}$ $(\beta_1+\beta_2I)$ if and only if $r(\alpha_1+\alpha_2I)=r(\beta_1+\beta_2I)$ ([22], Vol. 1, p. 57, Exercise 6). For each $r=1,\dots,n$, we denote the $\mathcal{J}$-class corresponding to r as J_r, so

$J_r=\{(\alpha_1+\alpha_2I)\in L(V){:}\,r(\alpha_1+\alpha_2I)=r\}.$

Thus, J_n equals G(V), the group of all nonsingular elements of L(V). According to [23] Lemma 3.2, every $(\alpha_1+\alpha_2 I)\in J_n-1$ can be expressed as a product of idempotents, each with rank n - 1. Our initial result extends this concept to $1\le r\le n-2$, T(X), the semigroup of all total transformations of a set X, where $|X|=n$ ([26] Lemma 5). The proof of this result draws heavily from that of [29] Theorem 3.

Theorem 1. *For $1\le r\le n-1$, every $(\alpha_1+\alpha_2 I)\in J_r$ can be expressed as a product of idempotents with in J_r.*

Theorem 2. *Let J be an neutrosophic idempotent matrix and $r\le n-1$. If $2r\le n$, then $(\alpha_1+\alpha_2 I)$ can be expressed as the product of two nilpotent neutrosophic matrices with rankr and index 2. However, if $2r>n$, we can write $r=q(n-r)+s$, where $q\ge 1$ and $0\le s\le n-r$. In this case, $(\alpha_1+\alpha_2 I)$ can be expressed as the product of two nilpotent neutrosophic matrices with rank r, each with index $q+1$ if $s=0$, or $q+2$ if $s>0$.*

Proof: Let $ker(\alpha_1+\alpha_2 I)=\{(e_{11}+e_{21}I),\dots,(e_{n-r}+e_{n-r}I)\}$ and $ran(\alpha_1+\alpha_2 I)=\{(a_{11}+a_{21}I))\dots(a_{1r}+a_{2r}I)\}$. If $2r\le n$ (hence $r\le n-r$) the required decomposition is:

$$(\alpha_1+\alpha_2 I)=\begin{pmatrix}\{E_{1,n-r}\} & (a_{11}+a_{21}I)) & \dots & (a_{1r}+a_{2r}I)\\ 0 & (a_{11}+a_{21}I) & \dots & (a_{1r}+a_{2r}I)\end{pmatrix}$$
$$=\begin{pmatrix}\{E_{1,r}\} & \{E_{r+1,n-r}\} & (a_{11}+a_{21}I) & \dots & (a_{1r}+a_{2r}I)\\ 0 & 0 & (e_{11}+e_{21}I) & \dots & (e_{1r}+e_{2r}I)\end{pmatrix}\circ$$
$$\begin{pmatrix}\{A_{1,r}\} & \{E_{r+1,n-r}\} & (e_{11}+e_{21}I), & \dots, & (e_{1r}+e_{2r}I)\\ 0 & 0 & (a_{11}+a_{21}I) & \dots & (a_{1r}+a_{2r}I)\end{pmatrix} \tag{3}$$

where, $E_{i,j}=(e_{1i}+e_{2i}I),\dots,(e_{1j}+e_{2j}I)$ and similarly, $A_{i,j}=(a_{1i}+a_{2i}I),\dots,(a_{1j}+a_{2j}I)$ However,we write $(\alpha_1+\alpha_2 I)=(\lambda_1+\lambda_2 I)(\mu_1+\mu_2 I)$ if $r=q(n-r)+s$ and $0\le s<n-r$,where $(\lambda_1+\lambda_2 I),(\mu_1+\mu_2 I)$ are defined as follows (and we use $m=n-r$ as an abbreviation):

$$(\lambda_1+\lambda_2 I)=\begin{pmatrix} \{E_{1,m}\} & 0 \\ (a_{11}+a_{21}I) & (e_{11}+e_{21}I) \\ \vdots & \\ (a_{1m}+a_{2m}I) & (e_{1m}+e_{2m}I) \\ (a_{1(m+1)}+a_{2(m+1)}I & (a_{11}+a_{21}I) \\ \vdots & \vdots \\ (a_{2m}+a_{2m}I) & (a_{1m}+a_{2m}I) \\ \vdots & \vdots \\ (a_{1(q-1)m+1}+a_{2(q-1)m+1}I) & (a_{1(q-2)m+1}+a_{2(q-2)m+1}I) \\ \vdots & \vdots \\ (a_{1(q-1)m+s}+a_{2(q-1)m+s}I) & (a_{1(q-2)m+s}+a_{2(q-2)m+s}I) \\ \vdots & \vdots \\ (a_{1qm}+a_{2qm}I) & (a_{1(q-1)m}+a_{2(q-1)m}I) \\ (a_{1(qm+1}+a_{2(qm+1)}I) & (a_{1(q-1)m+1}+a_{2(q-1)m+1}I) \\ (a_{1(qm+s)}+a_{2(qm+s)}I) & (a_{1(q-1)m+s}+a_{2(q-1)m+s}I) \end{pmatrix}^T \tag{4}$$

$$(\mu_1+\mu_2 I)=\begin{pmatrix} A_{(q-1)m+s+1,qm+s} & 0 \\ (e_{11}+e_{21}I) & (a_{11}+a_{21}I) \\ (a_{1m}+a_{2m}I) & (e_{1m}+e_{2m}I) \\ (a_{11}+a_{21}I) & (a_{1(m+1)}+a_{2(m+1)}I \\ \vdots & \vdots \\ (a_{1m}+a_{2m}I) & (a_{2m}+a_{2m}I) \\ \vdots & \vdots \\ (a_{1(q-2)m+1}+a_{2(q-2)m+1}I) & (a_{1(q-1)m+1}+a_{2(q-1)m+1}I) \\ \vdots & \vdots \\ (a_{1(q-2)m+s}+a_{2(q-2)m+s}I) & (a_{1(q-1)m+s}+a_{2(q-1)m+s}I) \\ \vdots & \vdots \\ (a_{1(q-1)m}+a_{2(q-1)m}I) & (a_{1qm}+a_{2qm}I) \\ (a_{1(q-1)m+1}+a_{2(q-1)m+1}I) & (a_{1(qm+1}+a_{2(qm+1)}I) \\ (a_{1(q-1)m+s}+a_{2(q-1)m+s}I) & (a_{1(qm+s)}+a_{2(qm+s)}I) \end{pmatrix}^T \tag{5}$$

It has been verified that each of these nilpotent neutrosophic elements has an index of $q+1$ or $q+2$, depending on whether $s=0$ or $s>0$.

We assert that in the aforementioned context, the indices $q+1$ and $q+2$ are optimal when $2r>n$. Specifically, if $(\alpha_1+\alpha_2 I)\in J_r$ is a product of nilpotent neutrosophic elements in J_r, then their indices must be at least $q+1$ or $q+2$, contingent upon whether $n-r$ divides r or not. To elaborate, we cite the following result from [28], Theorem 11.10.

Theorem 3. *If* $(B_1 + B_2 I)$ *is an* $n \times n$ *nilpotent neutrosophic matrix, then there exists a nonsingular neutrosophic matrix* $(P_1 + P_2 I)$ *such that*
$(P_1 + P_2 I)(B_1 + B_2 I)(P_1 + P_2 I)^{-1} = diag(H_{11} + H_{21} I, H_{21} + H_{22} I, \dots, H_{1k} + H_{2K} I)$
where each $H_{1i} + H_{2i} I$ $1 \le i \le k$ is an $n \times n$ upper triangular Jordan matrix with zeros on its main diagonal. In particular, the index of $B_1 + B_2 I$ equals the order of the largest $H_{1i} + H_{2i} I$.

In this way, let $(B_1 + B_2 I)$ be a $n \times n$ nonzero nilpotent neutrosophic matrix . Using the previous notation, the rank of each $H_{1i} + H_{2i} I$ equals 0 (if $n_i = 1$) or$n_i - 1$ (if$n_i > 1$). Suppose $H_{11} + H_{21} I, H_{12} + H_{22} I, \dots, H_{1k} + H_{2k} I$ are the nonzero $(H_{1i} + H_{2i} I)'s$. Then,

$$\sum_{i=1}^{k} (n_i - 1) = r = r(B_1 + B_2 I) \le n - r.$$

that is, $k \le (n - r)$ Furthermore, if the index of each $(H_{1i} + H_{2i} I)$ is at most q, then $r(H_{1i} + H_{2i} I) \le q - 1$ for $1 \le i \le k$, and we obtain

$$r = r(B_1 + B_2 I) \le \sum_{i=1}^{k} (q - 1) \le (q - 1) \cdot (n - r).$$

which is a contradiction if $r = q(n - r)$.that is if $r = q(n - r)$, then there must exist some $H_{1i} + H_{2i} I$ such that its index is at least $q + 1$. Similarly, if $r = q(n - r) + s$ and $0 < s < n - r$, then there must exist some $H_{1i} + H_{2i} I$ such that its index is at least $q + 2$.

Suppose $A_1 + A_2 I$ is an $n \times n$ nutrosophic matrix with rank $r \le n - 1$, which can be expressed as a product of nilpotent neutrosophic matrices $B_{1j} + B_{2j} I$, each with rank r. Based on the earlier remark, consider the following: If $r = q(n - r)$, then according to Theorem 3, the Jordan form of $B_{1j} + B_{2j} I$ must contain some block with index at least $q + 1$. Consequently, the index of each $B_{1j} + B_{2j} I$ in this case is at least $q + 1$.

Similarly, if $r = q(n - r) + s$ where $0 < s < n - r$ each nilpotent neutrosophic matrix $B_{1j} + B_{2j} I$ must have an index of at least $q + 2$. Therefore, the indices $q + 1$ and $q + 2$ in Theorem 2 are shown to be the best possible values under these conditions.

As discussed in Section 1, the structure of the semigroup generated by nilpotent matrices in $P_1(X) + P_2(X) I$ was established in [32], and its characterization depends on whether X contains an even or odd number of elements. Interestingly, this dependence does not arise in the context of $L(V)$ when $\dim V = n < \mathcal{N}_0$.

Corollary 1. *If* V *has dimension* $n < \mathcal{N}_0$*, then an element* $(\alpha_1 + \alpha_2 I) \in L(V)$ *is a product of neutrosophic nilpotent matrices in* $L(V)$ *if and only if the rank of* $\alpha_1 + \alpha_2 I$ *is less than* n.

Incidentally, Hannah and O'Meara [25] demonstrated that in any regular ring, every nilpotent element can be expressed as a product of idempotent elements. Consequently, in the regular ring $L(V)$, every nilpotent element belongs to $L_{n-1} = \{\alpha_1 + \alpha_2 I \in L(V): rank(\alpha_1 + \alpha_2 I) \le n - 1\}$, indicating $N(V) \subseteq L_{n-1}$. $N(V) = L_{n-1}$, Corollary 1.

However, it's important to note that Hannah and O'Meara's result relies on a significant theorem from the theory of regular rings. This method of proving Corollary 1 does not specify the precise rank and index conditions of the nilpotent elements required for the decomposition of an $(\alpha_1 + \alpha_2 I) \in L_{n-1}$.

Corollary 2. *Every finite neutrosophic semigroup* $N(S)$ *containting* n *can be embedded into a regular nilpotent-generated semigroup formed by linear transformations of an* $(n+1)$*-dimensional vector space.*

Proof: Let $N(S)^1$ (if $N(S)N(S)^1$ as a basis for an $(n+1)$ -dimensional vector space v. Write $N(S) = \{(s_{11} + s_{21}I), \dots, (s_{1n} + s_{2n}I)\}$. $t \in N(S)$, $p_t \in L(V)$

$$p_t = \begin{pmatrix} (s_{11} + s_{21}I) & (s_{12} + s_{22}I) & \dots & (s_{1n} + s_{2n}I) & 1 \\ 0 & (s_{12} + s_{22}I)t & \dots & (s_{1n} + s_{2n}I)t & t \end{pmatrix} \quad (6)$$

Clearly, $kerp_t$ contains $\langle(s_{11} + s_{21}I)\rangle$, the subspace generated by t, so $p_t \in N(V)$ (the nilpotent elements of $L(V)$ by Corollary 1. , $\langle N(V), \{p_t\}_{t\in S}\rangle$ is the required embedding.

Lemma 2. *If* $(\alpha_1 + \alpha_2 I)$, $(\beta_1 + \beta_2 I) \in L(V)$ *where* $dimV = n < \mathcal{N}_0$, *then* $rank((\alpha_1 + \alpha_2 I)(\beta_1 + \beta_2 I)) \geq rank(\alpha_1 + \alpha_2 I) + rank((\beta_1 + \beta_2 I) - n)$.

$\{(c_{11} + c_{21}I), \dots, (c_{1r} + c_{2r}I)\}$ $ran((\alpha_1 + \alpha_2 I)(\beta_1 + \beta_2 I))$, $(a_{1i} + a_{2i}I)(\alpha_1 + \alpha_2 I) = (b_{1i} + b_{2i}I)$ $(\beta_1 + \beta_2 I)(b_{1i} + b_{2i}I) = (c_{1i} + c_{2i})$ $1 \leq i \leq r$.
$(ran((\alpha_1 + \alpha_2 I)(\beta_1 + \beta_2 I))$ $(ran((\beta_1 + \beta_2 I)))$, $(\dim W = s)$.
$(span(\{(b_{11} + b_{21}I), \dots, (b_{1r} + b_{2r}\}))$ $(ran((\alpha_1 + \alpha_2 I)))$, $(\dim Q = t$. $(n = r + s + p)$ $n = \dim V$ $n((\beta_1 + \beta_2 I)) = p$.

Also, if $W = \langle\{(d_{11} + d_{21}I), \dots, (d_{1s} + d_{2s}\}\rangle$ and $(e_{1j} + e_{2j}I)(\beta_1 + \beta_2 I) = (e_{1j} + e_{2j}I)$ for$1 \leq j \leq s$ then

$$Q \cap \langle\{(e_{11} + e_{21}I), \dots, (e_{1s} + e_{2s}I), (b_{11} + b_{21}I), \dots, (b_{1r} + b_{2r}I)\}\rangle = 0$$

if $(u_1 + u_2 I) \in Q$ satisfies $(\alpha_1 + \alpha_2 I)(u_1 + u_2 I) = \sum (x_{1j} + x_{2j}I)(d_{1j} + d_{2j}I) + \sum (y_{1i} + y_{2i}I)(b_{1i} + b_{2i}I)$ for some $(u_1 + u_2 I) \in Q$ and some scalars $(x_{1j} + x_{2j}I)$ and $(y_{1i} + y_{2i}I)$, then $\sum (x_{1j} + x_{2j}I)(d_{1j} + d_{2j}I) \in ran((\alpha_1 + \alpha_2 I)(\beta_1 + \beta_2 I))$, $(x_{1j} + x_{2j}I) = 0$ j. ,$(u_1 + u_2 I) \in span(\{(b_{11} + b_{21}I), \dots, (b_{1r} + b_{2r}\})$, $(u_1 + u_2 I) = 0$. ,$n \geq r + s + 1$, and so $p \geq t$.

$$\begin{aligned} &rank((\alpha_1 + \alpha_2 I)) + rank((\beta_1 + \beta_2 I)) - n \\ &= (r + p) + (s + p) - n \\ &= r + 1 - p \leq r \\ &= rank((\alpha_1 + \alpha_2 I)(\beta_1 + \beta_2 I)). \end{aligned}$$

$(\alpha_1 + \alpha_2 I) \in L(V)$, let $(i(\alpha_1 + \alpha_2 I))$ $(\alpha_1 + \alpha_2 I)$. [24], $\dim V = n$ $(\alpha_1 + \alpha_2 I) \in L(V)$ $i(\alpha_1 + \alpha_2 I) \leq n$: ([28],).

Theorem 4. *If* $a \in L(V)$ *is a nilpotent operator with rank* r, *then* $i(\alpha_1 + \alpha_2 I) \geq \frac{n}{n-r}$. *In particular, any nilpotent operator in* J_n *has index* n.

The previous result contrasts sharply with the infinite-dimensional case as shown in ([33], Theorem 3.3), where nilpotent operators with index 2 are adequate to generate $N(V)$.

6. Conclusion

In conclusion, the study of real nilpotent neutrosophic matrices reveals intriguing insights into ma-trix theory and its applications in handling indeterminate data. The analysis of their important properties, such as rank, index, and linear transformations, underscores their utility in modeling uncertainty and ambiguity. The multiplication of two real nilpotent neutrosophic matrices demon-strates complex interactions where indeterminacy propagates and impacts the resultant matrix structure.

Moreover, exploring Green's relations and their implications within semigroups highlights how neutrosophic logic can enrich our understanding of algebraic structures. These matrices provide a nuanced approach to representing and manipulating uncertain information, offering a flexible toolfor decision-making in various fields, including economics, engineering, and artificial intelligence.

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