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Short communication

The Diophantine Equation $7^x + 147^y = z^2$

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Abstract. In this paper, we show that the Diophantine equation $7^x + 147^y = z^2$ has exactly two non-negative integer solutions (x, y, z), which are (2, 1, 14) and (5, 2, 196).

Keywords: Diophantine equation; Congruence; Non-negative integer solution

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1. Introduction

In the past decade, many researchers have been interested in studying and finding non-negative integer solutions (x, y, z) to the Diophantine equation in the form $7^x + n^y = z^2$, where n is a positive integer. For examples of research: in 2011, Suvarnamani, Singta and Chotchaisthit [1] showed that the Diophantine equation $4^x + 7^y = z^2$ has no non-negative integer solution. In 2013, Sroysang [2, 3] proved that the Diophantine equation $5^x + 7^y = z^2$ has no non-negative integer solution and $(x, y, z) = (0,1,3)$ is the unique non-negative integer solution of the Diophantine equation $7^x + 8^y = z^2$. After that, in 2014, Sroysang [4, 5] proved that the Diophantine equations $7^x + 19^y = z^2$, $7^x + 91^y = z^2$ and $7^x + 31^y = z^2$ have no non-negative integer solution. In 2018, Rao [6] showed that the Diophantine equation $3^x + 7^y = z^2$ has exactly two non-negative integer solutions (x, y, z), which are $(1,0,2)$ and $(2,1,4)$. In the same year, Burshtein [7] solved the Diophantine equation $2^{2x+1} + 7^y = z^2$, where x, y and z are positive integers and y is even. In 2019, Burshtein [8] established that the Diophantine equation $7^x + 10^y = z^2$ has no positive integer solution. In 2019, Asthana and Singh [9] showed that the Diophantine equation $2^x + 7^y = z^2$ has exactly three non-negative integer solutions (x, y, z). The solutions are $(3,0,3)$, $(5,2,9)$ and $(1,1,3)$. In 2020, Burshtein [10] studied positive integer solutions of the Diophantine equation $7^x + 11^y = z^2$. In 2022, Pakapongpun and Chattae [11] examined the non-negative integer solutions of the Diophantine equation $p^x + 7^y = z^2$, where p is a prime number. In 2022, Borah and Dutta [12] showed that

$(x, y, z) = (2, 1, 9)$ is the unique non-negative integer solution of the Diophantine equation $7^x + 32^y = z^2$. In 2024, Singh [13] showed that the Diophantine equation $7^x + 17^y = z^2$ has no positive integer solution. Recently, in 2025, Raksangoen, Tongnuy and Tadee [14] studied and proved that the Diophantine equation $7^x + 35^y = z^2$ has exactly one non-negative integer solution $(x, y, z) = (0, 1, 6)$.

From the study of the above researches, we are interested in finding the solutions of the Diophantine equation $7^x + 147^y = z^2$, where x, y and z are non-negative integers, by using elementary methods and Nam's Theorems, which will be discussed in the next topic.

2. Preliminaries

In the beginning of this section, we present some theorems, which were proved by Nam in 2024 [15].

Theorem 2.1. [15] Let n be a positive integer. The Diophantine equation $(3^n)^x + 1 = z^2$ has only solution is $(x, z) = (1, 2)$ if $n = 1$ and has no non-negative integer solution if $n > 1$.

Theorem 2.2. [15] Let m be a positive integer such that $4^m + 3$ is prime. Then the Diophantine equation $1 + (4^m + 3)^y = z^2$ has no non-negative integer solution.

Theorem 2.3. [15] Let $x = 2t$ be even for some positive integer t. Then the Diophantine equation $(3^n)^x + 7^y = z^2$ has only positive integer solution is $(x, y, z) = (2, 1, 4)$ if $n = 1$, and has no solution if $n > 1$.

Moreover, we show a property of congruence that will help to find solutions of the Diophantine equation $7^x + 147^y = z^2$.

Theorem 2.4. Let y be a non-negative integer. If y is odd, then $3^y \equiv 3, 5, 6 \pmod 7$.

Proof: Since y is odd, we get $y = 6k + i$ for some non-negative integer k and $i \in \{1, 3, 5\}$.

Case 1. $i = 1$. Then $3^y = 3^{6k+1} = 3 \cdot (3^6)^k \equiv 3 \cdot 1^k \equiv 3 \pmod 7$.

Case 2. $i = 3$. Then $3^y = 3^{6k+3} = 3^3 \cdot (3^6)^k \equiv 6 \cdot 1^k \equiv 6 \pmod 7$.

Case 3. $i = 5$. Then $3^y = 3^{6k+5} = 3^5 \cdot (3^6)^k \equiv 5 \cdot 1^k \equiv 5 \pmod 7$.

From the above three cases, we can conclude that $3^y \equiv 3, 5, 6 \pmod 7$.

3. Main results

In this section, we present our results.

The Diophantine Equation $7^x + 147^y = z^2$

Theorem 3.1. All non-negative integer solutions (x,y,z) of the Diophantine equation $7^x+147^y=z^2$ are $(2,1,14)$ and $(5,2,196)$.

Proof: Let x,y and z be non-negative integers such that $7^x+147^y=z^2$ or $7^x+3^y\cdot 7^{2y}=z^2$. Since $7^x+147^y\equiv(-1)^x+(-1)^y\pmod 4$ and $z^2\equiv 0,1\pmod 4$, we obtain that x and y have opposite parity. Assume that $y=0$. Then $1+7^x=z^2$. This is impossible, by Theorem 2.2. Thus $y>0$. Next, we consider the following cases:

Case 1. $x<2y$. Then $7^x\left(1+3^y\cdot 7^{2y-x}\right)=z^2$. Since $\gcd\left(7^x,1+3^y\cdot 7^{2y-x}\right)=1$, it implies that x is even and so y is odd. There exists a non-negative integer k such that $x=2k$. Let $w=\frac{z}{7^k}$. Therefore, $1+3^y\cdot 7^{2y-x}=w^2$ or $(w-1)(w+1)=3^y\cdot 7^{2y-x}$.

Case 1.1. $w-1=1$ and $w+1=3^y\cdot 7^{2y-x}$. Then $2=3^y\cdot 7^{2y-x}-1$ or $3=3^y\cdot 7^{2y-x}$. Since $x<2y$, we get $7\mid 3$. This is impossible.

Case 1.2. $w-1=3^y$ and $w+1=7^{2y-x}$. It follows that $2=7^{2y-x}-3^y$. This is impossible since $7^{2y-x}-3^y\equiv 1-0\equiv 1\pmod 3$.

Case 1.3. $w-1=7^{2y-x}$ and $w+1=3^y$. It follows that $2=3^y-7^{2y-x}$. This is impossible since $3^y-7^{2y-x}\equiv 3^y\equiv 3,5,6\pmod 7$, by Theorem 2.4.

Case 1.4. $w-1=3^y\cdot 7^{2y-x}$ and $w+1=1$. Since $w-1<w+1$, we get $3^y\cdot 7^{2y-x}<1$. This is impossible.

Case 2. $x=2y$. Therefore $3^y+1=\left(\frac{z}{7^y}\right)^2$. By Theorem 2.1, we have $\left(y,\frac{z}{7^y}\right)=(1,2)$. Then $y=1, z=14$ and so $x=2$. That is $(x,y,z)=(2,1,14)$.

Case 3. $x>2y$. Then $7^{2y}\left(7^{x-2y}+3^y\right)=z^2$ or $3^y+7^{x-2y}=\left(\frac{z}{7^y}\right)^2$. Assume that y is odd. Then, x is even. There exists a non-negative integer k such that $x=2k$. It follows that $\left(\frac{z}{7^y}\right)^2-\left(7^{k-y}\right)^2=3^y$ or $\left(\frac{z}{7^y}-7^{k-y}\right)\left(\frac{z}{7^y}+7^{k-y}\right)=3^y$. Since 3 is a prime number, we get $2\cdot 7^{k-y}=3^y-1$. Therefore $3^y-1=2\cdot 7^{k-y}\equiv 0\pmod 7$ and so $3^y\equiv 1\pmod 7$. This is impossible, by Theorem 2.4. Thus, y is even. By Theorem 2.3, it implies that $\left(y,x-2y,\frac{z}{7^y}\right)=(2,1,4)$. Thus $y=2, x=5$ and $z=196$. That is $(x,y,z)=(5,2,196)$.

Hence, $(2,1,14)$ and $(5,2,196)$ are all non-negative integer solutions (x,y,z) of the Diophantine equation $7^x+147^y=z^2$.

From Theorem 3.1, we prove the following corollaries.

Corollary 3.2. The Diophantine equation $49^x + 147^y = z^2$ has the unique non-negative integer solution $(x, y, z) = (1, 1, 4)$.

Proof: Let x, y and z be non-negative integers such that $49^x + 147^y = z^2$ or $7^{2x} + 147^y = z^2$. By Theorem 3.1, we consider the following cases:

Case 1. $(2x, y, z) = (2, 1, 14)$. It implies that $(x, y, z) = (1, 1, 14)$.

Case 2. $(2x, y, z) = (5, 2, 196)$. Then $2x = 5$. This is impossible.

Hence, $(x, y, z) = (1, 1, 14)$ is the unique non-negative integer solution of the Diophantine equation $49^x + 147^y = z^2$.

Corollary 3.3. Let n be a positive integer and x, y, z be non-negative integers. Then all solutions (n, x, y, z) of the Diophantine equation $7^x + 147^y = z^{2n}$ are $(1, 2, 1, 14)$, $(1, 5, 2, 196)$ and $(2, 5, 2, 14)$.

Proof: Let n be a positive integer and x, y, z be non-negative integers such that $7^x + 147^y = z^{2n}$ or $7^x + 147^y = (z^n)^2$. By Theorem 3.1, we consider the following cases:

Case 1. $(x, y, z^n) = (2, 1, 14)$. Therefore $x = 2, y = 1$ and $z^n = 14$. It implies that $(n, x, y, z) = (1, 2, 1, 14)$.

Case 2. $(x, y, z^n) = (5, 2, 196)$. Therefore $x = 5, y = 2$ and $z^n = 196$. It implies that $(n, x, y, z) \in \{(1, 5, 2, 196), (2, 5, 2, 14)\}$.

Hence, $(1, 2, 1, 14)$, $(1, 5, 2, 196)$ and $(2, 5, 2, 14)$ are all solutions (n, x, y, z) of the Diophantine equation $7^x + 147^y = z^{2n}$.

4. Conclusion

By using some properties of congruence and Nam's Theorems, we prove that the Diophantine equation $7^x + 147^y = z^2$ has exactly two non-negative integer solutions (x, y, z), which are $(2, 1, 14)$ and $(5, 2, 196)$. An interesting thing to study further is the search for all solutions of Diophantine equation $7^x + 3^y \cdot 7^z = w^2$, when x, y, z and w are non-negative integers.

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Conflict of interest. The paper is written by a single author so there is no conflict of interest.

Authors' Contributions. It is a single-author paper. So, full credit goes to the author.

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