Annals of Pure and Applied Mathematics
Vol. 32, No. 1, 2025, 7-16
ISSN: 2279-087X (P), 2279-0888(online)
Published on 6 August 2025
www.researchmathsci.org
DOI: http://dx.doi.org/10.22457/apam.v32n1a02969

Annals of
Pure and Applied Mathematics

Approximation Properties of a New Baskakov-Kantorovich Type Operators

Ahmed F. Jabber[1*] **and *Amal K. Hassan***[2]

[1,2]Department of Mathematics, College of Scenic, University of Basrah, Basrah, Iraq
Email: amal.khelil@uobasrah.edu.iq
*Corresponding author: Pgs.ahmed.fadhil @uobasrah.edu.iq

Received 29 June 2025; accepted 4 August 2025

Abstract. In this paper, we construct three new kinds of Baskakov-Kantorovich operators. We discuss a uniform convergence estimate for these operators. Also, some direct results concerning the order of convergence and asymptotic formulas of the new operator are given. Finally, we illustrate the convergence of the operators to a certain function with the help of Maple software.

Keywords: Baskakov operators, Kantorovich Operators, Voronovskaja Theorem.

AMS Mathematics Subject Classification (2020): 41A10, 41A25, 41A36.

1. Introduction

The classical Kantorovich Operators ([2] and [5]) of bounded continuous functions $f(x)$ on the interval $[0,\infty)$, which are defined as:

$$L_n(f;x) = (\mathrm{n}-1)\sum_{k=0}^{\infty} P_{n,k}(x) \int_{\frac{k}{n-1}}^{\frac{k+1}{n-1}} f(t)dt \ , \tag{1.1}$$

where,

$$P_{n,k}(x) = \frac{(n+k-1)!\ \ x^k}{k!\,(n-1)!\ \ (1+x)^{n+k}}.$$

Wafi and Khatoon [14] proved a Voronovskaya-type theorem for generalised Baskakov-Kantorovich operators in polynomial weight spaces. Wafi and Rao [13], we introduced generalized Baskakov-Kantorovich operators based on two parameters and proved an order of approximation using second modulus of continuity, Ditzian-Totik modulus of smoothness, Peeter's K-functional. In [1], Deniz et al. constructed Baskakov-Durrmeyer Kantorovich operators in terms of the method of Stan constructed Bernstein-Durrmeyer-Kantorovich operators in [12]. Agrawal [6], studied the bivariate extension of the generalized Baskakov-Kantorovich operators and discussed the comparison of the convergence of the bivariate generalized Baskakov Kantorovich operators and the bivariate Szasz-Kantorovich operators. Erencin and Buyukdurakoglu [3], we introduced some direct results and weighted approximation properties for a modi cation of generalized Baskakov-Kantorovich operators. Gupta [7], we proposed three new operators, which are obtained from composition of genealized Baskakov-Szasz operators with Szasz and Lupas operators based on Laguerre polynomials. Jabbar and Hassan ([8, 9]) have introduced a modified Baskakov operators to improve the degree approximation as follows :

$$B_n^1(f;x) = \sum_{k=0}^{\infty} \mathcal{M}_{n,k}(x)\, f\left(\frac{k}{n}\right), \tag{1.2}$$

$$\mathcal{M}_{n,k}(x) = \psi(x,n)P_{n+1,k}(x) + \psi(-1-x,n)P_{n+1,k-1}(x), \tag{1.3}$$

and

$$\psi(x,n) = a(n) + b(n)x, n = 0,1 \ldots,$$

where $a(n)$ and $b(n)$ are unknow sequences. For $a(n) = b(n) = 1$, obviously (1.2) reduces to classical Baskakov operators. The main aim of this research is to present a new construction of the Baskakov-Kantorovich operator, which has better features and properties than the usual Baskakov-Kantorovich operator.

2. Kantorovich operators of order I

For $\gamma > 0$ and $f \in C_\gamma(I) = \{f \in C(I): |f(x)| \le C(1 + x^\gamma), \forall\, x \in I\}$, where $I = [0,\infty)$ and C is a positive constant dependent on f. For any $f \in C_\gamma(I)$ we define a Kantorovich operator as

$$H_{n,1}(f;x) = (n-1)\sum_{k=0}^{\infty} \mathcal{M}_{n,k}(x) \int_{\frac{k}{n-1}}^{\frac{k+1}{n-1}} f(t)dt \ . \tag{2.1}$$

Lemma 2.1. The following statements hold

i) $H_{n,1}(f;x) = 2a(n) - b(n)$

ii) $H_{n,1}(t;x) = \frac{1}{2(n-1)}\Big(2nx\big(2a(n) - b(n)\big) + 4x\big(a(n) - b(n)\big) + 4a(n) - 3b(n)\Big)$

iii) $H_{n,1}(t^2;x) = \frac{1}{3(n-1)^2}\Big(3\,n^2x^2\big(2a(n) - b(n)\big) + 3nx^2\big(6a(n) - 5b(n)\big) + 6nx\big(3a(n) - 2b(n)\big) + 12x^2\big(a(n) - b(n)\big) + 18x\big(a(n) - b(n)\big) + 8a(n) - 7b(n)\Big)$

In order to study the uniform convergence, we consider the sequences $a(n)$ and $b(n)$. Verify the condition

$$2a(n) - b(n) = 1. \tag{2.2}$$

We will consider two cases for the sequences $a(n)$ and $b(n)$.

Case 1: Let

$$a(n) - b(n) \ge 0 \text{ and } a(n) \ge 0. \tag{2.3}$$

Using (2.2), we get $0 \leq a(n) \leq 1$ and $-1 \leq b(n) \leq 1$. In this case, the operator (2.2) is positive.

Case 2: Let

$$a(n) - b(n) < 0 \text{ or } a(n) < 0. \tag{2.4}$$

If $a(n) - b(n) < 0$, then $a(n) > 1$ and if $a(n) < 0$, then $a(n) - b(n) > 1$. In this case, the operator (2.1) is non-positive.

Theorem 2.2. Let $a(n), b(n)$ be two sequence which verify the conditions (2.2) and (2.3). If $f \in C_\gamma(I)$, then

$$\lim_{n\to\infty} H_{n,1}(f;x) = f(x),$$

uniformly on $[u, v] \subset [0, \infty)$.

Proof: In view of Lemma 2.1 it is obvious that $H_{n,1}(t^i;x) \to x^i, i = 0,1.2$ when $n \to$ $, which$ gives the prove of Theorem (2.2). □

Theorem 2.3. [11] Let $h \in C[u,v] > 0$ be a function and suppose that $(L_n)_{n\geq 1}$ is a sequence of positive Linear operators such that for any $\lim_{n\to\infty} L_n(e_n) = he_i, i = 0,1,2,$ uniformly on $[u,v]$. Then for a given function $f \in C[u,v], we$ have $\lim_{n\to\infty} L_n(f) = hf$ uniformly on $[u,v]$.

Theorem 2.4. Let $f \in C_\gamma(I)$, then for all convergence sequences $a(n)$ and $b(n)$ that verify conditions (2.2) and (2.4), we have

$$\lim_{n\to\infty} H_{n,1}(f;x) = f(x),$$

uniformly on $[u, v] \subset [0, \infty)$.

Proof: From (2.2) and (2.4), it follows that

$$(a(n) > 0, b(n) > 0) \text{ or } (a(n) < 0, b(n) < 0).$$

Suppose that $a(n) > 0, b(n) > 0$. The operator $H_{n,1}(f;x)$ can be written as follows

$$H_{n,1}(f;x) = U_{n,1}(f;x) - W_{n,1}(f;x),$$

where

$$U_{n,1}(f;x) = (n-1)\sum_{k=0}^{\infty}\big((2a(n) + b(n)x)P_{n+1,k}(x) + a(n)P_{n+1,k-1}\big) \int_{\frac{k}{n-1}}^{\frac{k+1}{n-1}} f(t)dt$$

$$W_{n,1}(f;x) = (n-1)\sum_{k=0}^{\infty}\big(a(n)P_{n+1,k}(x) + \big(b(n)x + b(n)\big)P_{n+1,k-1}\big) \int_{\frac{k}{n-1}}^{\frac{k+1}{n-1}} f(t)dt.$$

Using Theorem 2.3, we obtain

$$\lim_{n\to\infty} U_{n,1}(f;x) = \left(L_2\left(x + \frac{3}{2}\right) + \frac{3}{2}\right) f(x),$$

$$\lim_{n\to\infty} W_{n,1}(f;x) = \left(L_2\left(x + \frac{3}{2}\right) + \frac{1}{2}\right) f(x),$$

where $\lim_{n\to\infty} b(n) = L_2$. Therefore, $\lim_{n\to\infty} H_{n,1}(f;x) = f(x)$. □

Lemma 2.5. The moments of the operators (2.1) are given by

(i) $\lim_{n\to\infty} nH_{n,1}(t-x;x) = \frac{1}{2}(1+2x)(2-L_2)$

(ii) $\lim_{n\to\infty} nH_{n,1}((t-x)^2;x) = x(1+x)$

(iii) $\lim_{n\to\infty} n^2H_{n,1}((t-x)^4;x) = 3\,x^2(1+x)^2$

Theorem 2.6. Let $a(n)$ and $b(n)$ be a convergent sequence that satisfies the conditions (2.2) and (2.3). If $f'' \in C_\gamma(I)$, then

$$\lim_{n\to\infty} n\left(H_{n,1}(f;x) - f(x)\right) = \left(\frac{1}{2}(1+2x)(2-L_2)\right)f'(x) + \frac{1}{2}x\,(1+x)f''(x).$$

Proof: Using the Cauchy-Schwarz inequality and Lemma (2.5), we get the desired result. □

3. Kantorovich Operators of order II

In this section, we will extend the previous results considering the modified Kantorovich-type operators as follows

$$H_{n,2}(f;x) = (n-1)\sum_{k=0}^{\infty} \mathcal{B}_{n,k}(x) \int_{\frac{k}{n-1}}^{\frac{k+1}{n-1}} f(t)dt \ , \tag{3.1}$$

where

$$\mathcal{B}_{n,k}(x) = \hat{W}(x,n)P_{n+2,k}(x) + \hat{U}(x,n)P_{n+2,k-1}(x) + \hat{W}(-1-x,n)P_{n+2,k-2}(x), \tag{3.2}$$

and moreover

$$\hat{W}(x,n) = c(n) + d(n)x + e(n)x^2,\ \ \hat{U}(x,n) = g(n)(x+x^2)\,, \tag{3.3}$$

where $c(n), d(n), e(n)$ and $g(n)$ are some unknown sequences which are determined in appropriate way. For $c(n) = e(n) = 1, d(n) = -g(n) = 2$ obviously, (3.1) reduces to (1.1).

Since,

$$H_{n,2}(1;x) = 2e(n)x^2 + g(n)x^2 + 2e(n)x + g(n)x + 2c(n) - d(n) + e(n).$$

Therefore, to study the rete of convergence we set $H_{n,2}(1;u) = 1$, and these yields

$$2c(n) - d(n) + e(n) = 1, \tag{3.4}$$

$$g(n) + 2e(n) = 0. \tag{3.5}$$

Since,

$$H_{n,2}(t;x) = \tfrac{1}{2(n-1)}\Big(nx^3\big(4e(n)+2g(n)\big) + nx^2\big(4e(n)+2g(n)\big) + \big(8e(n) + 4g(n)\big) + 2nx\big(2c(n)-d(n)+e(n)\big) + x^2\big(14e(n)+7g(n)\big) + 8x\big(1-c(n)\big) + 5 - 4c(n)\Big).$$

To prove $H_{n,2}(t;x)$, we consider the sequences $c(n)$ to verify the condition

$$\lim_{n\to\infty} \frac{c(n)}{n} = 0.$$

Using the equations (2.5) and (2,6), we obtain

$$H_{n,2}(t;x) = x + \frac{(1+2x)\left(5-4c(n)\right)}{2(n-1)}.$$

We propose $c(n) = \frac{5}{4}$. Hence,

$$H_{n,2}(t;x) = x.$$

In order to have $H_{n,2}(t^2;x)$ we consider the sequence $e(n)$ to verify the condition

$$\lim_{n\to\infty} \frac{e(n)}{n^2} = 0.$$

Using the equations above, we get

$$H_{n,2}(t^2;x) = x^2 + \frac{3n(x+x^2) - 21(x+x^2) + 6e(n)(x+x^2) - \frac{7}{2}}{3(n-1)^2}$$

We put, $e(n) = \frac{-n}{2}$. Therefore,

$$H_{n,2}(t^2;x) = x^2 - \frac{7(1+6x+6x^2)}{6(n-1)^2}.$$

With the above choices, the operator (2.1) becomes

$$H_{n,2}(f;x) = (n-1)\sum_{k=0}^{\infty} \mathcal{B}_{n,k}(x) \int_{\frac{k}{n-1}}^{\frac{k+1}{n-1}} f(t)dt, \qquad (3.6)$$

where

$$\mathcal{B}_{n,k}(x) = \left(\frac{5}{4} + \left(\frac{3-n}{2}\right)x - \frac{n}{2}x^2\right)P_{n+2,k}(x) + n(x+x^2)P_{n+2,k-1}(x) - \left(\frac{1}{4} + \left(\frac{3+n}{2}\right)x + \frac{n}{2}x^2\right)P_{n+2,k-2}(x)$$

Lemma 3.1. The moments of the operators (3.1) are given by

(i) $H_{n,2}(t-x;x) = 0$

(ii) $H_{n,2}((t-x)^2;x) = -\frac{7(1+6x+6x^2)}{6(n-1)^2}$

(iii) $H_{n,2}((t-x)^3;x) = -\frac{28\,nx^3+42\,nx^2+14\,nx+200\,x^3+300\,x^2+130\,x+15}{4(n-1)^3}$

(iv) $H_{n,2}((t-x)^4;x) = -(3x^4+6x^3+3x^2)\frac{n^2}{(n-1)^4} + O\left(\frac{1}{n^3}\right)$

(v) $H_{n,2}((t-x)^5;x) = -(90x^5+225x^4+180x^3+45x^2)\frac{n^2}{(n-1)^5} + O\left(\frac{1}{n^4}\right)$

(vi) $H_{n,2}((t-x)^6;x) = -(30x^6+90x^5+90x^4+30x^3)\frac{n^3}{(n-1)^6} + O\left(\frac{1}{n^4}\right)$

Theorem 3.2. Let $f^{(4)} \in C_\gamma(I)$. Then

$$\lim_{n\to\infty} n^2\left(H_{n,2}(f;x)-f(x)\right) = \frac{-7(1+6x+6x^2)}{12}\,f''(x) - \frac{(28x^3+42x^2+14x)}{24}f^{(3)}(x) - \frac{(3x^4+6x^3+3x^2)}{24}f^{(4)}(x).$$

Proof : Applying the operator $H_{n,2}(f;x)$ to the Taylor's formula, we obtain

$$H_{n,2}(f;x)-f(x) = H_{n,2}\big((t-x);x\big)f'(x)+\frac{1}{2}H_{n,2}((t-x)^2;x)f''(x) + \frac{1}{6}H_{n,2}((t-x)^3;x)f^{(3)}(x)+\frac{1}{24}H_{n,2}((t-x)^4;x)f^{(4)}(x) + H_{n,2}\,(\,\theta(t,x)(t-x)^4;x),$$

where $\theta(t,x)\longrightarrow 0$ as $t\longrightarrow x$.
From the Lemma (2.1), we have

$$\lim_{n\to\infty} n^2\left(H_{n,2}(f;x)-f(x)\right) = \frac{-7(1+6x+6x^2)}{12}\,f''(x) - \frac{(28x^3+42x^2+14x)}{24}f^{(3)}(x) - \frac{(3x^4+6x^3+3x^2)}{24}f^{(4)}(x) + \lim_{n\to\infty} n^2\,H_{n,2}\,(\,\theta(t,x)(t-x)^4;x).$$

Now, we want to show that $\lim_{n\to\infty} n^2\,H_{n,2}\,(\,\theta(t,x)(t-x)^4;x)=0.$
Here, we cannot use Cauchy – Schwarz inequality because $H_{n,2}(f;x)$ is not a positive operator in this case, we use the following technique:
For given $\varepsilon>0$ there is $\delta>0$ such that if $|\,t-x|<\delta\rightarrow|\theta(t,x)|<\varepsilon$ and for $|\,t-x|\geq\delta$, then there is a constant $C>0$ such that $|\theta(t,x)|<C$.

$$n^2\left|H_{n,2}\,(\,\theta(t,x)(t-x)^4;x)\right| \leq n^2(n-1)\sum_{k=0}^{\infty}\mathcal{B}_{n,k}(x)\,(x)\int_{|\,t-x|<\delta}|\theta(t,x)|\,(t-x)^4dt + n^2(n-1)\sum_{k=0}^{\infty}\mathcal{B}_{n,k}(x)\int_{|\,t-x|\geq\delta}|\theta(t,x)|\,(t-x)^4dt\ ,$$

$$\leq n^2(n-1)\varepsilon\sum_{k=0}^{\infty}\mathcal{B}_{n,k}(x)\int_{\frac{k}{n-1}}^{\frac{k+1}{n-1}}(t-x)^4f(t)\,dt + \frac{C\,n^2(n-1)}{\delta^2}\sum_{k=0}^{\infty}\mathcal{B}_{n,k}(x)\int_{\frac{k}{n-1}}^{\frac{k+1}{n-1}}(t-x)^6f(t)\,dt$$

$$\leq\ n^2(n-1)\varepsilon\,O\left(\frac{1}{n^3}\right)+\frac{n^2(n-1)C}{\delta^2}\,O\left(\frac{1}{n^4}\right)=o(1).$$

The proof of this theorem is complete. □

4. Kantorovich operators of order III

Using the approach as in the previous section we construct a third order approximation formula. Let us consider the following modification of Kantorovich Operators.

$$H_{n,3}(f;x) = (n-1)\sum_{k=0}^{\infty} \mathcal{D}_{n,k}(x) \int_{\frac{k}{n-1}}^{\frac{k+1}{n-1}} f(t)dt, \tag{4.1}$$

where,

$$\mathcal{D}_{n,k}(x) = \phi(x,n)P_{n+4,k}(x) + \lambda(x,n)P_{n+4,k-1}(x) + \theta(x,n)P_{n+4,k-2}(x) + \lambda(-1-x,n)P_{n+4,k-3}(x) + \phi(-1-x,n)P_{n+4,k-4}(x), \tag{4.2}$$

and moreover

$$\phi(x,n) = a_0(n) + a_1(n)x + a_2(n)x^2 + a_3(n)x^3 + a_4(n)x^4, \tag{4.3}$$

$$\lambda(x,n) = b_0(n) + b_1(n)x + b_2(n)x^2 + b_3(n)x^3 + b_4(n)x^4, \tag{4.4}$$

$$\theta(x,n) = c_0(n)(x+x^2)^2. \tag{4.5}$$

We note that $a_i(n), a_i(n), i = 0, \dots, 4$, and $c_0(n)$ are some unidentified sequences that are identified in this manner. For $a_0(n) = a_4(n) = 1, a_1(n) = a_3(n) = 4, a_2(n) = 6, b_0(n) = 0, b_1(n) = b_4(n) = -4, b_2(n) = b_3(n) = -12$ and $c_0(n) = 6$ obviously, (3.1) reduces to (1.1).

Since,

$$H_{n,3}(1;x) = 2a_4(n)x^4 + 2b_4(n)x^4 + c_0(n)x^4 + 4a_4(n)x^3 + 4b_4(n)x^3 + 2c_0(n)x^3 + 2a_2(n)x^2 - 3a_3(n)x^3 + 6a_4(n)x^4 + 2b_2(n)x^2 - 3b_3(n)x^3 + 6b_4(n)x^4 + c_0(n)x^2 + 2a_2(n)x - 3a_3(n)x + 4a_4(n)x + 2b_2(n)x - 3b_3(n)x + 4b_4(n)x + 2a_0(n) - a_1(n) + a_2(n) - a_3(n) + a_4(n) + 2b_0(n) - b_1(n) + b_2(n) - b_3(n) + b_4(n).$$

Therefore, to study the rete of convergence we set $H_{n,3}(1;x) = 1$, and these yields

$$2\big(a_0(n) + b_0(n)\big) - \big(a_1(n) + b_1(n)\big) + \big(a_2(n) + b_2(n)\big) - \big(a_3(n) + b_3(n)\big) + \big(a_4(n) + b_4(n)\big) = 1, \tag{4.6}$$

$$2\big(a_2(n) + b_2(n)\big) - 3\big(a_3(n) + b_3(n)\big) + 4\big(a_4(n) + b_4(n)\big) = 0, \tag{4.7}$$

$$c_0(n) + 2\big(a_4(n) + b_4(n)\big) = 0. \tag{4.8}$$

Since,

$$H_{n,3}(t;x) = \frac{1}{2(n-1)}\Big(2nx^5\big(2a_4(n) + 2b_4(n) + c_0(n)\big) + 2nx^4\big(4a_4(n) + 4b_4(n) + 2c_0(n)\big) + 2x^5\big(8a_4(n) + 8b_4(n) + 4c_0(n)\big) + 2nx^3\big(2a_2(n) - 3a_3(n) + 6a_4(n) + 2b_2(n) - 3b_3(n) + 6b_4(n) + c_0(n)\big) + x^4\big(42a_4(n) + 42b_4(n) + 21c_0(n)\big) + 2nx^2\big(2a_2(n) - 3a_3(n) + 4a_4(n) + 2b_2(n) - 3b_3(n) + 4b_4(n)\big) + x^3\big(16a_2(n) - 32a_3(n) + 84a_4(n) + 16b_2(n) - 28b_3(n) + 76b_4(n) + 18c_0(n)\big) + 2nx\Big(2\big(a_0(n) + b_0(n)\big) - \big(a_1(n) + b_1(n)\big) + \big(a_2(n) + b_2(n)\big) - \big(a_3(n) + b_3(n)\big) + \big(a_4(n) + b_4(n)\big)\Big) + x^2\big(26a_2(n) - 51a_3(n) + 86a_4(n) + 26b_2(n) - 45b_3(n) + 74b_4(n) + 5c_0(n)\big) + x\big(16a_0(n) - 16(n) + 26a_2(n) -$$

$35\, a_3(n)\ + 44\, a_4(n)\ + 16b_0(n) - 12b_1(n) + 22\, b_2(n)\ - 29b_3(n)\ + 36b_4(n)\big) +$
$10a_0(n) - 9a_1(n) + 9a_2(n) - 9a_3(n) + 9a_4(n) + 10b_0(n) - 7b_1(n) + 7b_2(n) -$
$7b_3(n) + 7b_4(n)\Big).$

Hence, to proof $D_n(t;x)$,we consider the sequences $a_i(n)$ and $b_i(n), i = 0, \dots ,4$ to verify the conditions

$$\lim_{n\to\infty}\frac{a_0(n)}{n} = 0,\ \lim_{n\to\infty}\frac{b_0(n)}{n} = 0 \quad \text{and}$$
$$2\big(2a_4(n) + b_4(n)\big) - \big(2a_3(n) + b_3(n)\big) = 0. \tag{4.9}$$

Using the equations (4.6), (4.7), (4.8) and (4.9), we obtain

$$H_{n,3}(t;x) = \frac{2nx + \left(\frac{170}{9} - 16a_0(n) - 8b_0(n)\right)x + \left(\frac{94}{9} - 8a_0(n) - 4b_0(n)\right)}{2(n-1)}.$$

We put, $a_0(n) = \frac{137}{72}$ and $b_0(n) = \frac{-43}{36}$.

Therefore,

$$H_{n,3}(t;x) = x.$$

In order to have $H_{n,3}(t^2;x)$ we consider the sequences $a_i(n)$ and $b_4(n), i = 1, ,2,3,4$ to verify the conditions

$$\lim_{n\to\infty}\frac{a_1(n)}{n} = 0\ ,\ \lim_{n\to\infty}\frac{a_2(n)}{n^3} = 0,\ \lim_{n\to\infty}\frac{a_3(n)}{n^3} = 0 \quad \text{and}$$
$$4a_4(n) + b_4(n) = 0. \tag{4.10}$$

Using the equations (4.6), (4.7), (4.8),(4.9) and (4,10), we get

$$H_{n,3}(t^2;x) = x^2 + \frac{(3n-138)(x+x^2) + (x+x^2)(36a_1(n) + 9a_3(n) - 18a_2(n))}{3(n-1)^2}$$

We propose our analysis for the cases,

$$a_1(n) = \frac{69}{8} - \frac{17}{24}n,\ a_2(n) = \frac{n^2}{8} \ \text{and}\ a_3(n) = \frac{-115}{6} + \frac{5}{2}n + \frac{n^2}{4}. \tag{4.11}$$

Therefore,

$$H_{n,3}(t^2;x) = x^2.$$

By using equation (4.11), we have

$$a_4(n) = -\frac{101}{8} + \frac{43}{24}n + n^2,\ b_1(n) = -\frac{45}{4} + \frac{5}{4}n,\ b_2(n) = \frac{115}{4} - \frac{15}{4}n - \frac{1}{2}n^2\ ,$$
$$b_3(n) = \frac{533}{6} - \frac{73}{6}n - n^2 \ \text{and}\ b_4(n) = \frac{101}{2} - \frac{43}{6}n - \frac{1}{2}n^2\ .$$

Lemma 4.1. The moments of the operators (4.1) are given by

(i) $H_{n,3}(t-x;x) = H_{n,3}((t-x)^2;x) = H_{n,3}((t-x)^3;x) = 0$

(ii) $H_{n,3}((t-x)^4;x) = (135x^4 + 270x^3 + 158x^2 + 23x)\frac{n}{(n-1)^4} + O\left(\frac{1}{n^4}\right)$

(iii) $H_{n,3}((t-x)^5;x) = \left(55x^5 + \frac{275}{2}x^4 + 110x^3 + \frac{55}{2}x^2\right)\frac{n^2}{(n-1)^5} + O\left(\frac{1}{n^4}\right)$

(iv) $H_{n,3}((t-x)^6;x) = (15\,x^6 + 45\,x^5 + 45\,x^4 + 15\,x^3)\frac{n^3}{(n-1)^6} + O\left(\frac{1}{n^4}\right)$

(v) $H_{n,3}((t-x)^7;x) = H_{n,3}((t-x)^8;x) = O\left(\frac{1}{n^4}\right)$

Theorem 4.2. Let $f^{(6)} \in C_\gamma(I)$.Then

$$\lim_{n\to\infty} n^3 \left(H_{n,3}(f;x) - f(x)\right) = O\left(\frac{1}{n^3}\right). \tag{4.12}$$

The proof can be concluded similar to that of theorem (3.2). □

5. Numerical results

The theoretical findings from the earlier sections will be analyzed in this section using numerical examples.

Example 1. Let $f(x) = \frac{Cos(2\pi x)}{e^x}$.
Example 2. Let $h(x) = Sin(4\pi x) + Sin(6x)$.
Example 3. Let $g(x) = \frac{Sin(5\pi x)}{4} + \frac{Cos(4\pi x)}{5}$.

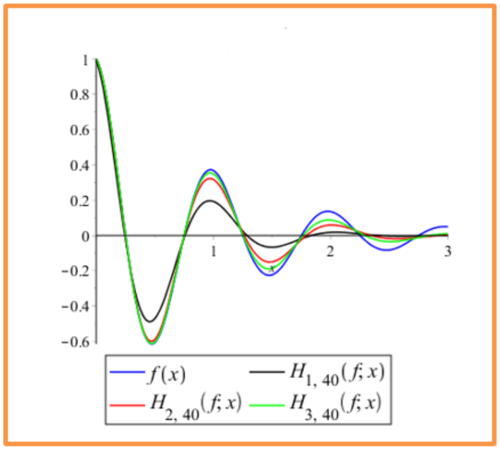

Figure 1:

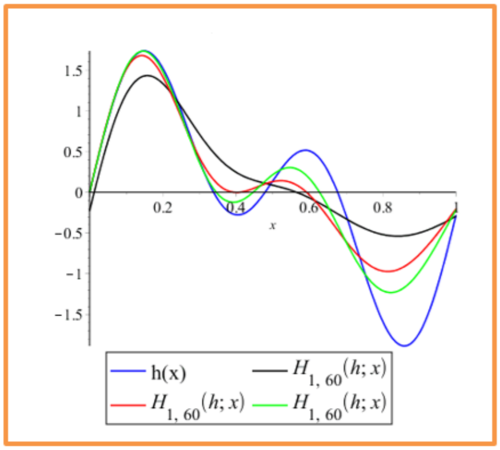

Figure 2:

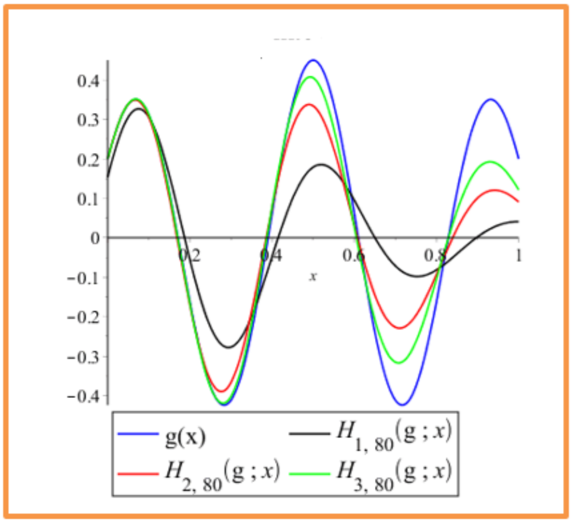

Figure 3:

Acknowledgements. The author expresses sincere gratitude to the anonymous referee for their valuable comments, constructive feedback, and insightful suggestions, which have significantly contributed to improving the clarity, quality, and overall presentation of this paper.

Author's Contributions: All authors have contributed equally to the research and preparation of this work.

Conflicts of interest: The authors declare no conflicts of interest.

REFERENCES

1. E. Deniz, A. Aral and G. Ulusoy, New integral type operators, *Filomat*, 31(9) (2017) 2851-2865.
2. Z. Ditzian and V. Totik, Moduli of smoothness, Springer, Berlin, (1987).
3. A. Erencin and S. Buyukdurakoglu, A modification of generalized Baskakov-Kantorovich operators, *Stud. Univ. Babes-Bolyai Math.*, 59(3) (2014) 351-364.
4. Mogoi N. Evans and Robert Obogi, Advances in the geometry of banach spaces: novel inequalities, operator theory and applications, *Annals of Pure and Applied Mathematics*, 31(2) (2025) 69-87.
5. Ivan Gadjev, Approximation of functions by Baskakov-Kantorovich operator, *Results in Mathematics*, (2016), https://doi.org/10.1007/s00025-016-0554-7
6. M. Goyal and P.N. Agrawal, Approximation properties of bivariate generalized Baskakov-Kantorovich operators, *Int. J. Nonlinear Anal. Appl.*, 14(10) (2023) 361-375, http://dx.doi.org/10.22075/ijnna.2023.29597.4203
7. V. Gupta, New discrete operators from generalized Baskakov operators, *Filomat*, 38(21) (2024) 7493-7501.
8. A. Jabbar and A. Hassan, Better Approximation Properties by New Modified Baskakov operators, *Journal of Applied Mathematics*, Article ID 8693699, 9 pages, https://doi.org/10.1155/2024/8693699
9. A. Jabbar and A. Hassan, Approximation by New Baskakov operators, *Gulf Journal of Mathematics*, 19(2) (2025) 442-450.
10. M.Kavitha, K.Rameshwar and P. Murugadas, Convergence and reduction of transitive neutrosophic soft matrices, *Annals of Pure and Applied Mathematics*, 30(2) (2024) 137-148.
11. H. Khosravian-Arab, M. Dehghan and M.R. Eslahchi, A new approach to improve the order of approximation of the Bernstein operators, *Theory and Applications*, *Numer Algorithms*, 77(1) (2018) 111-150.
12. I.G. Stan, On the Durrmeyer-Kantorovich type operator, *Bull. Transilvania Univer. Brasov, Series III: Math. Informat. Phys.*, 6(5) (2013) 33-42.
13. A. Wafi, N.Rao and D. Rai, Approximation Properties by Generalised-Baskakov Kantorovich-Stancu type operators, *Appl. Math. Inf. Sci. Lett.*, 4(3) (2018) 111-118.
14. A. Wafi and S. Khatoon, The Voronovskaya theorem for generalized Baskakov-Kantorovich operators in polynomial weight spaces, *Mat. Vesnik*, 57(3) (2005) 87-94.