Annals of Pure and Applied Mathematics
Vol. 32, No. 1, 2025, 17-26
ISSN: 2279-087X (P), 2279-0888(online)
Published on 9 August 2025
www.researchmathsci.org
DOI: http://dx.doi.org/10.22457/apam.v32n1a03970

Annals of
Pure and Applied
Mathematics

Implicit Finite-Difference Time-Stepping Method for Time-Fractional Fokker-Planck Equation

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Received 21 June 2025; accepted 2 August 2025

Abstract. We propose and analyse a numerical solution that is based on a time stepping Crank-Nicolson (C-N) combined with finite element (FE) in the space of the time fractional diffusion advection problem. We numerically implement the C-N solution and explore the possibility of modifying the scheme to address the issues arising from the presence of the fractional derivative. For the stability and the error analysis, we use some properties of the Riemann–Liouville fractional derivative operator, which are listed in the last section. For numerical simulations, we write an efficient computer code using MATLAB that illustrates numerically the convergence rates of the proposed C-N schemes on various model problems.

Keywords: Numerical solution, Implementation of numerical scheme, fractional diffusion, error bound

AMS Mathematics Subject Classification (2020): 35R70, 47F05

1. Introduction

Our fractional PDE is of the form:

$$\partial_t u - \nabla \cdot (\kappa \nabla \partial_t^{1-\alpha} u - F \partial_t^{1-\alpha} u) = g \tag{1}$$

We discretize in time the model problem. To do so, we let $0 = t_0 < t_1 < t_2 < \cdots < t_N = T$ and we use a time graded mesh with the following nodes $t_i = (i \ \ k)^\gamma$ for $0 \leq i \leq N$ with $\gamma \geq 1$ and $k = T^{1/\gamma}/N$, where N is the number of subintervals. Denote by $k_n = t_n - t_{n-1}$ the length of the nth subinterval $I_n = (t_{n-1}, t_n)$, for $1 \leq n \leq N$. In our notation, we will often suppress the dependence on x and think of $u = u(x,t)$ as a function of t taking values in $L_2(\Omega)$. Integrating the fractional Fokker–Planck equation (1) over the nth time interval I_n gives

$$u(t_n) - u(t_{n-1}) + \int_{I_n} \partial_t^{1-\alpha} \mathcal{A} u \ \ dt = \int_{I_n} g(x,t) \ \ dt. \tag{2}$$

where

$$\mathcal{A} u = -\nabla^2 u + \nabla \cdot (Fu)$$

We seek to compute $U^n(x) \approx u(x, t_n)$ for $n = 1, 2, , N$ by requiring that

$$U^n - U^{n-1} + \int_{I_n} \partial_t^{1-\alpha} \mathcal{A} \bar{U} \ \ dt = k_n \bar{g}^n \tag{3}$$

with $\bar{F}^n(x) = F(x, t_{n-\frac{1}{2}}), t_{n-\frac{1}{2}} = \frac{t_n + t_{n-1}}{2}$ and $\bar{U} = \frac{U^n + U^{n-1}}{2}$,

$\bar{g}^n \approx k_n^{-1} \int_{I_n} g(x,t) \; dt.$

The time stepping starts from the initial condition

$$U^0(x) = u_0(x) \qquad \text{for} 0 \le x \le L,$$

2. Previous work

When $F = 0$, numerical methods for (1) were proposed and analyzed by several authors. For time-stepping methods, Langlands [6] proposed backward Euler scheme for discretization the fractional derivative. Mclean and Mustapha [9] applied finite-difference time discretization combined with finite elements in space. For the discontinuous Galerkin in time and finite elements in space, we refer to Mclean and Mustapha [10]. Mustapha [12] investigated an implicit finite-difference Crank-Nicolson scheme combined with finite elements in space. For piecewise constant discontinuous Galerkin method to discretize the time, see McLean and Mustapha [11]. Later on Mustapha et al. [13] proposed and analyzed a time-stepping Petrov–Galerkin method combined with the continuous conforming finite elements method in space.

Sweilan et al. [14] proposed a Crank-Nicolson finite difference method to solve the linear time-fractional diffusion equation, formulated with Caputo's fractional derivative. Zeng et al. [15] developed a new Crank–Nicolson finite elements method in which a novel time discretization called the modified $L1$ method was used to discretize the Riemann–Liouville fractional derivative.

For *space discretisation, Zhang et al. [16] considered a standard central difference approximation for the spatial discretisation, for the time stepping, two new alternating direction implicit schemes based on the L1 approximation and backwards* Euler method were proposed to solve a two-dimensional anomalous sub-diffusion equation with time fractional derivative. For semidiscrete spatial finite volume method to approximate solutions of anomalous subdiffusion equations in a two-dimensional convex domain, we refer to Karaa et al. [4]. Jin et al. [3] applied Galerkin finite elements method and lumped mass Galerkin, using piecewise linear functions to solve initial boundary value problem for a homogeneous time-fractional diffusion equation in a bounded convex polygonal domain. Karaa et al. [5] applied a piecewise-linear finite elements method to approximate the solution of time-fractional diffusion equations on bounded convex domains. For general fractional convection-diffusion equation,

$$\mathcal{J}^{1-\alpha} u' - (a u_x)_x + b u_x + c u = f, \tag{4}$$

with coefficients a, b, c that may depend on x and t, Cui [1] investigated a high-order approximation for the time-fractional derivative combined with a compact exponential finite difference scheme for approximating the convection and diffusion terms.

Recently, Gracia et al. [2] applied a standard finite difference method on a uniform mesh to solve (1). They proved that the rate of convergence of the maximum nodal error on any subdomain that is bounded away from t = 0 is higher than the rate obtained when the maximum nodal error is measured over the entire space-time domain.

- Case of *space-time dependent* forcing F in *one space dimension*. Le et al. [7] proposed and analysed a piecewise-linear Galerkin finite elements method in space and an implicit Euler method for time to solve (1).
- For the case of the space-time dependent forcing F in *multi-dimensional space*.

Le et al. [8] presented a new stability and convergence analysis for the spatial discretization of (17) in a convex polyhedral domain, using continuous, piecewise-linear, finite elements. Their analysis used a novel sequence of energy arguments in combination with a generalized Gronwall inequality.

Definition 2.1 *[Ritz Projection Property]*

$$\| \rho^n \| \le Ch^2 \| u(t_n) \|_{H^2(\Omega)} \qquad \text{for} 0 \le \text{n} \le \text{N}. \tag{5}$$

3. An implicit Crank-Nicolson time-stepping scheme

We discretize in time the model problem (1). To do so, we let $0 = t_0 < t_1 < t_2 < \cdots < t_N = T$ and we use a time graded mesh with the following nodes $t_i = (i \;\; k)^\gamma$ for $0 \le i \le N$ with $\gamma \ge 1$ and $k = T^{1/\gamma}/N$, where N is the number of subintervals. Denote by $k_n = t_n - t_{n-1}$ the length of the nth subinterval $I_n = (t_{n-1}, t_n)$, for $1 \le n \le N$. In our notation, we will often suppress the dependence on x and think of $u = u(x,t)$ as a function of t taking values in $L_2(\Omega)$. Integrating the fractional Fokker–Planck equation (1) over the nth time interval I_n gives

$$u(t_n) - u(t_{n-1}) + \int_{I_n} \partial_t^{1-\alpha} \mathcal{A} u \;\; dt = \int_{I_n} g(x,t) \;\; dt. \tag{6}$$

where

$$\mathcal{A}u = -\nabla^2 u + \nabla \cdot (Fu)$$

We seek to compute $U^n(x) \approx u(x, t_n)$ for $n = 1,\ 2, ,\ N$ by requiring that

$$U^n - U^{n-1} + \int_{I_n} \partial_t^{1-\alpha} \mathcal{A} \bar{U} \;\; dt = k_n \bar{g}^n \tag{7}$$

with $\bar{F}^n(x) = F(x, t_{n-\frac{1}{2}}), t_{n-\frac{1}{2}} = \frac{t_n + t_{n-1}}{2}$ and $\bar{U} = \frac{U^n + U^{n-1}}{2}$,

$\bar{g}^n \approx k_n^{-1} \int_{I_n} g(x,t) \;\; dt.$

The time stepping starts from the initial condition

$$U^0(x) = u_0(x) \qquad \text{for} 0 \le \text{x} \le \text{L}, \tag{8}$$

and is subject to the boundary conditions $U^n(x) = 0$ for $x \in \partial\Omega$ where $1 \le n \le N$.

4. Stability of the numerical solution

In this section, we show the stability of the semidiscrete approximate solution U of (7) in the following theorem.

Theorem 1. *Consider the implicit scheme (7). Assume that the driving force* $\vec{F} = \vec{F}(x)$ *satisfies that*

$$\nabla \cdot F \ge \frac{-2\|\nabla u\|^2}{u^2} \quad \text{on} \quad \Omega$$

then

$$\| U^n \| \le \| U^0 \| + 2 \sum_{j=1}^{n} \| \tilde{g}^j \|.$$

Proof. Taking the inner product of (7) with $\bar{U}^n$,

$$\langle U^n - U^{n-1}, \bar{U}^n \rangle + \int_{I_n} \langle \partial_t^{1-\alpha} \mathcal{A} \bar{U}(t), \bar{U}(t) \rangle \;\; dt = \langle \bar{g}^n, \bar{U}^n \rangle$$

where

$$\mathcal{A}\bar{U} = -\nabla^2 \bar{U} + \nabla \cdot (F\bar{U})$$

Now, using the given assumption on $\vec{F}$

$$\langle \mathcal{A}\bar{U}, \bar{U} \rangle = \langle -\nabla^2 \bar{U} + \nabla \cdot (F\bar{U}), \bar{U} \rangle = \| \nabla \bar{U} \|^2 - \langle F\bar{U}, \nabla \bar{U} \rangle \ge 0. \tag{9}$$

Let $U^{n^*} = \max_{0\le n\le N} \| U^n \|$. Summing the above equation from $n = 1$ to $n = n^*$ gives

$$\| U^{n^*} \|^2 - \| U^0 \|^2 + 2\int_0^{t_{n^*}} \langle \partial_t^{1-\alpha}\mathcal{A}\bar{U}(t), \bar{U}(t)\rangle \; dt = \sum_{n=0}^{n^*} \langle \bar{g}^n, U^n + U^{n-1}\rangle.$$

Using (9) it follows that

$$\| U^{n^*} \|^2 \le \| U^0 \|^2 + 2 \| U^{n^*} \| \sum_{n=0}^{n^*} \| \bar{g}^n \| \le \| U^{n^*} \| (\| U^0 \| + 2\sum_{n=0}^{n^*} \| \bar{g}^n \|$$

which implies the desired result.

5. Error bound from the time dicretization

In this section, we estimate the error $e^n = U^n - u(t_n)$ when U^n is given by:

$$U^n - U^{n-1} + \int_{I_n} \partial_t^{1-\alpha}(\mathcal{A}\bar{U}) \; dt = k_n \bar{g}^n \tag{10}$$

and u is the solution of

$$u' + (\partial_t^{1-\alpha}\mathcal{A}u) = g, \tag{11}$$

Integrating (1) from $t = t_{n-1}$ to $t = t_n$ shows that the exact solution u satisfies:

$$u(t_n) - u(t_{n-1}) + \int_{I_n} (\partial_t^{1-\alpha}\mathcal{A}u) \; dt = k_n \bar{g}^n.$$

Comparing this with (10), we observe that the error e^n satisfies:

$$e^n - e^{n-1} + \int_{I_n} (\partial_t^{1-\alpha}\mathcal{A}\bar{e}) \; dt = \eta^n \tag{12}$$

where

$$\eta^n = \int_{t_{n-1}}^{t_n} (\partial_t^{1-\alpha}\mathcal{A}(u - \bar{u})(t)) \; dt \tag{13}$$

since $e^0 = U^0 - u_0$, the stability result in Theorem 1 implies that

$$\| e^n \| \le \| U^0 - u_0 \| + 2\sum_{j=1}^{n} \| \eta^j \| \tag{14}$$

In the next theorem, we estimate the error from the time discretisation.

Theorem 2. (Convergence theorem) *Let u be the solution of the initial-value problem (1) and let U^n be the solution of the discrete-time scheme (10). Assume that the initial data $u_0 \in H^2(\Omega)$ and Assume that*

$$t^\alpha \| u'(t) \| + t^{1+\alpha} \| u''(t) \| \le Ct^{\eta-1} \qquad , \; 0 < \eta < \alpha + 2, \qquad 0 < t < T. \tag{15}$$

Then, for $1 \le n \le N$, we have

$$\| U_h^n - u(t_n) \| \le \| U_h^0 - u_0 \| + Ch^2 + C \times \begin{cases} k^{\gamma\alpha} & \text{if} 1 \le \gamma < \frac{\alpha+1}{\alpha} \\ k^{\alpha+1}\max(1, \log(t_n/t_1)) & \text{if} \gamma = \frac{\alpha+1}{\alpha} \\ k^{\alpha+1} & \text{if} \gamma > \frac{\alpha+1}{\alpha} \end{cases}$$

Proof: From [Mustapha, [14]] we have

$$\| \eta_1^j \|^2 \le C(k^{2\gamma(\sigma+\alpha)} + k^{2+\gamma(\alpha+\sigma)} k_j t_j^{\alpha+\sigma-1-2/\gamma} + k^4 k_j^2 t_j^{2(\alpha+\sigma-1-2/\gamma)}).$$

Therefore,

$$\frac{1}{k_j} \| \eta_1^j \|^2 \le C(\frac{1}{k_j} k^{2\gamma(\sigma+\alpha)} + k^{2+\gamma(\alpha+\sigma)} t_j^{\alpha+\sigma-1-2/\gamma} + k^4 k_j t_j^{2(\alpha+\sigma-1-2/\gamma)}).$$

Using

$$k_j = t_j - t_{j-1} = k^\gamma (j^\gamma - (j-1)^\gamma) = \gamma k^\gamma \int_{j-1}^{j} t^{\gamma-1} \; dt \ge \gamma k^\gamma (j-1)^{\gamma-1},$$

implies that

$$\frac{1}{k_j} \leq C_\gamma k^{-\gamma}(j-1)^{1-\gamma}.$$

$$\sum_{j=1}^{n} \left(\frac{1}{k_j} \parallel \eta_1^j \parallel^2\right) = \frac{1}{k_1} \parallel \eta_1^1 \parallel^2 + \sum_{j=2}^{n} \left(\frac{1}{k_j} \parallel \eta_1^j \parallel^2\right)$$
$$\leq \frac{1}{k_1} \parallel \eta_1^1 \parallel^2 + \sum_{j=2}^{n} C k^{\gamma(2(\sigma+\alpha)-1)}(j-1)^{1-\gamma}$$
$$+C\sum_{j=2}^{n} k^{2+\gamma(\alpha+\sigma)} t_j^{\alpha+\sigma-1-2/\gamma} + C\sum_{j=2}^{n} k^4 k_j t_j^{2(\alpha+\sigma-1-2/\gamma)}$$
$$\leq Ck^{\gamma(2(\alpha+\sigma)-1))} + (Ck^{\gamma(2(\sigma+\alpha)-1)})\sum_{j=2}^{n} (j-1)^{1-\gamma}$$
$$+Ck^{\gamma(2(\alpha+\sigma)-1)} \sum_{j=2}^{n} j^{\gamma(\alpha+\sigma)-\gamma-2} + C\sum_{j=2}^{n} k^4 \int_{I_j} t^{2(\alpha+\sigma-1-2/\gamma)}\ dt.$$

Now,

$$\sum_{j=2}^{n} k^4 \int_{I_j} t^{2(\alpha+\sigma-1-2/\gamma)}\ dt = k^4 \int_{t_2}^{t_n} t^{2(\alpha+\sigma-1-2/\gamma)}\ dt$$
$$= Ck^4 \times \begin{cases} t_2^{2(\alpha+\sigma)-1-4/\gamma} & \text{if} 2(\alpha+\sigma)-2-4/\gamma < -1 \\ \log(t_n/t_2) & \text{if} 2(\alpha+\sigma)-2-4/\gamma = -1 \\ t_n^{2(\alpha+\sigma)-1-4/\gamma} & \text{if} 2(\alpha+\sigma)-2-4/\gamma > -1 \end{cases}$$
$$= Ck^4 \times \begin{cases} t_2^{2(\alpha+\sigma)-1-4/\gamma} & \text{if} 1 \leq \gamma < 2/(\alpha+\sigma-1/2) \\ \log(t_n/t_2) & \text{if} \gamma = 2/(\alpha+\sigma-1/2) \\ t_n^{2(\alpha+\sigma)-1-4/\gamma} & \text{if} \gamma > 2/(\alpha+\sigma-1/2) \end{cases}$$
$$\leq C \times \begin{cases} k^{\gamma(2(\alpha+\sigma)-1)} & \text{if} 1 \leq \gamma < 2/(\alpha+\sigma-1/2) \\ k^4 \log(t_n/t_2) & \text{if} \gamma = 2/(\alpha+\sigma-1/2) \\ k^4 & \text{if} \gamma > 2/(\alpha+\sigma-1/2) \end{cases}$$

For the series in the term

$$k^{\gamma(2(\alpha+\sigma)-1)} \sum_{j=2}^{n} j^{\gamma(\alpha+\sigma)-\gamma-2}$$

it converges only if $2+\gamma-\gamma(\alpha+\sigma) > 1$ which implies

$$\begin{cases} \text{if} \alpha+\sigma < 1 \Longrightarrow \gamma > \frac{1}{\alpha+\sigma-1} \quad \text{which is true for any } \gamma \\ \text{if} \alpha+\sigma > 1 \Longrightarrow 1 \leq \gamma < 1/(\alpha+\sigma-1) \end{cases}$$

Therefore we can combine the above results as follows

$$k^{\gamma(2(\alpha+\sigma)-1)} \sum_{j=2}^{n} j^{\gamma(\alpha+\sigma)-\gamma-2} + C\sum_{j=2}^{n} k^4 \int_{I_j} t^{2(\alpha+\sigma-1-2/\gamma)}\ dt$$
$$\leq C \times \begin{cases} k^{\gamma(2(\alpha+\sigma)-1)} & \text{if} 1 \leq \gamma < 2/(\alpha+\sigma-1/2) \\ k^4 \log(t_n/t_2) & \text{if} \gamma = 2/(\alpha+\sigma-1/2) \\ k^4 & \text{if} \gamma > 2/(\alpha+\sigma-1/2) \end{cases}$$

For the case $(\alpha+\sigma) < 1/2$, it is contained by the first case since $\gamma > 2\alpha+\sigma-1/2$, as the right-hand side is negative and $\gamma \geq 1$. The first series on the left side is convergent by the integral test. Therefore,

$$\sum_{j=1}^{n} \left(\frac{1}{k_j} \parallel \eta_1^j \parallel^2\right) \leq C \times \begin{cases} k^{\gamma(2(\alpha+\sigma)-1)} & \text{if} 1 \leq \gamma < 2/(\alpha+\sigma-1/2) \\ k^4 \log(t_n/t_2) & \text{if} \gamma = 2/(\alpha+\sigma-1/2) \\ k^4 & \text{if} \gamma > 2/(\alpha+\sigma-1/2) \end{cases}$$

Using (5), followed by regularity assumption (15) one can conclude that

$$\parallel \eta_2^j \parallel^2 \leq Ch^4\left(\int_{I_j} \parallel u'(t) \parallel_2\ \ dt\right)^2 \leq Ch^4\left(\int_{I_j} t^{\sigma-1}\ \ dt\right)^2 \leq Ch^4 k_j^2 t_j^{2(\sigma-1)}$$

Hence,

$$\sum_{j=1}^{n} \left(\frac{1}{k_j}(\| \eta_2^j \|^2)\right) \le Ch^4 \sum_{j=1}^{n} \frac{1}{k_j} k_j^2 t_j^{2(\sigma-1)}$$
$$\le Ch^4 \sum_{j=1}^{n} k_j t_j^{2\sigma-2} \le Ch^4 \int_{t_1}^{t_n} t^{(2\sigma-1)-1} \, dt$$
$$\le Ch^4 \times \begin{cases} \log(t_n/t_1) & \text{if} \sigma = 1/2 \\ t_n^{2\sigma-1} & \text{if} \sigma > 1/2 \end{cases}$$

Combining the above estimates leads to:

$$\| \theta^n \|^2 \le t_n \sum_{j=1}^{n} \left(\frac{1}{k_j}(\| \eta_1^j \|^2 + \eta_2^j \|^2)\right) \le$$
$$Ct_n(h^4 k^{\gamma(2\sigma-1)} + \begin{cases} k^{\gamma(2(\alpha+\sigma)-1)} & \text{if} 1 \le \gamma < 2/(\alpha+\sigma-1/2) \\ k^4 \log(t_n/t_2) & \text{if} \gamma = 2/(\alpha+\sigma-1/2) \\ k^4 & \text{if} \gamma > 2/(\alpha+\sigma-1/2) \end{cases}$$

consequently,

$$\| \theta^n \| \le Ch^2 k^{\gamma(\sigma-1/2)} + C \times$$
$$\begin{cases} k^{\gamma(\alpha+\sigma-1/2)} & \text{if} 1 \le \gamma < 2/(\alpha+\sigma-1/2) \\ k^2 \max(1, \sqrt{\log(t_n/t_2)}) & \text{if} \gamma = 2/(\alpha+\sigma-1/2) \\ k^2 & \text{if} \gamma > 2/(\alpha+\sigma-1/2) \end{cases} .$$

Combining the above estimates,

$$\| u_h^n - u(t_n) \| = \| \theta^n + \rho^n \| \le Ch^2 + \| \theta^n \| + \| \rho^n \|$$
$$\le Ch^2 + C \times \begin{cases} k^{\gamma(\alpha+\sigma-1/2)} & \text{if} 1 \le \gamma < 2/(\alpha+\sigma-1/2) \\ k^2 \max(1, \sqrt{\log(t_n/t_2)}) & \text{if} \gamma = 2/(\alpha+\sigma-1/2) \\ k^2 & \text{if} \gamma > 2/(\alpha+\sigma-1/2) \end{cases}$$

for $1 \le n \le N$.

6. Implementation and numerical experiments

In Section 1, we discuss the implementation of the Crank-Nicolson finite elements scheme in one dimension. The implementation of the $L1$ approximation scheme is discussed in Section 2. The last section contained numerical experiments that confirm our theoretical convergence results for both numerical schemes. Some figures and numerical tables will be included.

6.1. Implementations of the Crank-Nicolson finite element scheme

Recall that our fully-discrete solution $U_h^n \in S_h$ is given by

$$\langle U_h^n - U_h^{n-1}, v \rangle + \int_{I_n} \langle \partial_t^{1-\alpha} \nabla \bar{U}_h, \nabla v \rangle \, dt - \int_{I_n} \langle \bar{F}^n \partial_t^{1-\alpha} \bar{U}_h, \nabla v \rangle \, dt = \int_{I_n} \langle g, v \rangle \, dt$$

for all $v \in S_h$ and for $1 \le n \le N$, with $U_h^0 = R_h u_0$. Explicitly, let $\phi_p \in S_h$ denote the pth nodal basis function so that $\phi_p(x_q) = \delta_{pq}$. So,

$$U_h^n(x) = \sum_{p=1}^{P-1} U_p^n \phi_p(x) \qquad \text{where} \mathrm{U_p^n = U_h^n(x_p) \approx U^n(x_p) \approx u(x_p, t_n)}.$$

Define the $(P-1) \times (P-1)$ tridiagonal matrices $\mathbf{M}$ and $\mathbf{B^n}$ with entries

$$\mathbf{M_{pq}} = \langle \phi_q, \phi_p \rangle \qquad \text{and} \qquad \mathbf{B^n_{pq}} = \langle \phi_{qx}, \phi_{px} \rangle - \langle \bar{F}^n \phi_q, \phi_{px} \rangle,$$

and define $(P-1)$-dimensional column vectors U^n and $\mathbf{G^n}$ with components $\mathbf{U_p^n}$ and $G_p^n = \int_{I_n} \langle g, \phi_p \rangle \, dt$. We find that

$$\mathbf{MU^n} - \mathbf{M}U^{n-1} + \frac{1}{2}\omega_{nn}\mathbf{B^nU^n} + \frac{1}{2}\sum_{j=1}^{n-1}\omega_{nj}\mathbf{B^nU^j} + \frac{1}{2}\omega_{nn}\mathbf{B^nU^{n-1}}$$

$$+\frac{1}{2}\sum_{j=1}^{n-1}\omega_{nj}\mathbf{B^nU^{j-1}} + \sum_{j=1}^{n}\omega_{nj}\mathbf{B^nU^j} - \sum_{j=1}^{n-1}\omega_{n-1,j}\mathbf{B^n}U^j - \sum_{j=1}^{n-1}\omega_{n-1,j}\mathbf{B^nU^{j-1}} = \mathbf{G^n}, \qquad (16)$$

where

$$\omega_{nj} = \int_{I_j}\omega_\alpha(t_n - s)\ \ ds = \omega_{1+\alpha}(t_n - t_{j-1}) - \omega_{1+\alpha}(t_n - t_j) \qquad \text{for} n \geq 2.$$

Therefore, at the nth time step, we must solve the following linear system

$$\left(\mathbf{M} + \frac{1}{2}\omega_{nn}\mathbf{B^n}\right)\mathbf{U^n} = \left(\mathbf{M} - \frac{1}{2}\omega_{nn}\mathbf{B^n}\right)\mathbf{U^{n-1}} + \mathbf{G^n} - \frac{1}{2}\sum_{j=1}^{n-1}(\omega_{nj} - \omega_{n-1,j})\mathbf{B^nU^j}$$

$$-\frac{1}{2}\sum_{j=1}^{n-1}(\omega_{nj} - \omega_{n-1,j})\mathbf{B^nU^{j-1}} \qquad (17)$$

with

$$M = \begin{bmatrix} \phi_{11} & \phi_{12} & \cdots & 0 & 0 \\ \phi_{21} & \phi_{22} & \phi_{23} & 0 \ddots & \vdots \\ 0 & \ddots & \ddots & \ddots & 0 \\ \vdots & \ddots & \phi_{p-2,p-3} & \phi_{p-2,p-2} & \phi_{p-2,p-1} \\ 0 & \cdots & 0 & \phi_{p-1,p-2} & \phi_{p-1,p-1} \end{bmatrix},$$

$$B^n = \begin{bmatrix} \psi_{11} & \psi_{12} & \cdots & 0 & 0 \\ \psi_{21} & \psi_{22} & \psi_{23} & \cdots & 0 \\ 0 & \ddots & \ddots & \ddots & 0 \\ \vdots & \ddots & \psi_{p-2,p-3} & \psi_{p-2,p-2} & \vdots \\ 0 & \cdots & 0 & \psi_{p-1,p-2} & \psi_{p-1,p-1} \end{bmatrix}$$

$$-\begin{bmatrix} \xi_{11} & \xi_{12} & \cdots & 0 & 0 \\ \xi_{21} & \xi_{22} & \xi_{23} & \cdots & 0 \\ 0 & \ddots & \ddots & \ddots & 0 \\ \vdots & \ddots & \xi_{p-2,p-3} & \xi_{p-2,p-2} & \vdots \\ 0 & \cdots & 0 & \xi_{p-1,p-2} & \xi_{p-1,p-1} \end{bmatrix}$$

where $\phi_{ij} = \langle\phi_i, \phi_j\rangle$, $\psi_{ij} = \langle\partial_x(\phi_i), \partial_x(\phi_j)\rangle$, $\xi_{ij} = \langle F^n\phi_i, \partial_x\phi_j\rangle$,

$$g_i = \int_{t_{n-1}}^{t_n}\langle g(t), \phi_i\rangle, \qquad 0 \leq i,j \leq p-1$$

$$U^n = \begin{bmatrix} U_1^n \\ U_2^n \\ \vdots \\ U_{P-1}^n \end{bmatrix}, \qquad G^n = \begin{bmatrix} g_1 \\ g_2 \\ \vdots \\ g_{p-1} \end{bmatrix}$$

7. Numerical convergence

The convergence of both numerical methods (Crank-Nicolson and $L1$) will be tested on a sample example below. Choose

$$F(x,t) = x + \sin t, \qquad T = 1, \qquad L = \pi, \qquad \kappa_\alpha = \mu_\alpha = 1,$$

where the source term g is chosen so that the exact solution $u(x,t) = [1 + \omega_{1+\alpha}(t)]\sin x$. In this example the solution u satisfies the following regularity properties:

$$t^{\alpha} \parallel u'(t) \parallel + t^{1+\alpha} \parallel u''(t) \parallel \leq Ct^{2\alpha-1}.$$

This valid for $\sigma = 2\alpha$. Hence, from the error analysis in chapter 4 we expect the convergence rate of the Crank-Nicolson finite elements scheme to be of order $O(k^{2\alpha\gamma})$ for $1 \leq \gamma < \frac{1+\alpha}{2\alpha}$ and $O(k^{1+\alpha})$ for $\gamma > \frac{1+\alpha}{2\alpha}$.

Whereas for the $L1$ approximation scheme, the required regularity assumption is

$$\parallel u'(t) \parallel + t^{2} \parallel u''(t) \parallel \leq t^{\sigma-1}$$

This is valid for $\sigma = \alpha$. Hence we expect $O(k^{\gamma(2\alpha-1/2)})$ rates of convergence for $1 \leq \gamma < 2/(2\alpha - 0.5)$ and $O(k^2)$ for $\gamma > 2/(2\alpha - 0.5)$. The numerical results in Table 1 show a better convergence rate.

$\alpha = 0.3$								
N	$\gamma = 1$				$\gamma = 2$			
	$Crank - Nicolson$		$L1$		$Crank - Nicolson$		$L1$	
	$M.E.$	$O.C.$	$M.E.$	$O.C.$	$M.E.$	$O.C.$	$M.E.$	$O.C.$
20	6.87e-02		1.72e-02		1.52e-02		4.46e-03	
40	4.96e-02	0.47	1.31e-02	0.39	7.22e-03	1.07	2.17e-03	1.03
80	3.53e-02	0.49	9.73e-03	0.43	3.34e-03	1.11	1.03e-03	1.08
160	2.48e-02	0.51	7.04e-03	0.47	1.52e-03	1.14	4.78e-04	1.11
320	1.73e-02	0.52	5.02e-03	0.49	6.79e-04	1.16	2.17e-04	1.14
640	1.2e-02	0.53	3.54e-03	0.51	3.01e-04	1.17	9.72e-05	1.16
Theory		0.6		0.1		1.2		0.2

$\alpha = 0.3$								
N	$\gamma = 3$				$\gamma = 3.3$			
	$Crank - Nicolson$		$L1$		$Crank - Nicolson$		$L1$	
	$M.E.$	$O.C.$	$M.E.$	$O.C.$	$M.E.$	$O.C.$	$M.E.$	$O.C.$
20	9.279e-03		9.08e-04		1.204e-02		5.5e-04	
40	4.067e-03	1.19	2.85e-04	1.67	4.493e-03	1.18	1.54e-04	1.83
80	1.74e-03	1.21	8.63e-05	1.72	1.95e-03	1.2	4.33e-05	1.83
160	7.51e-04	1.22	2.56e-05	1.75	8.4e-04	1.21	1.2e-05	1.85
320	3.19e-04	1.23	7.42e-06	1.78	3.55e-04	1.23	3.18e-06	1.92
640	1.34e-04	1.25	2.16e-06	1.78	1.5e-04	1.24	8.2e-07	1.96
Theory		1.3		0.3		1.3		2

Table 1: Errors and convergence rates for different mesh grading γ with $\alpha = 0.3$.

We observe better order for $L1$ scheme. The errors and convergence rates for Crank-Nicolson and $L1$ improved when the mesh is graded. We observe that the numerical results of Crank-Nicolson are as expected in Theorem (2). However, the numerical results of the

$L1$ scheme shows that the theoretical results are pessimistic.

8. Conclusion

We established the existence and uniqueness of the weak solution for the general form of the model problem (1) in the case of space-time dependent driving forcing via Galerkin method. Furthermore the behavior of the time derivatives of the weak solution was studied, proving estimates that play an important role in the error analysis of the numerical schemes. For the numerical solution of the model problem (1), an implicit Crank-Nicolson scheme to discretize in time was proposed such a scheme is formally second-order accurate. However, due to the presence of a weakly singular kernel and the fractional derivative operator $\partial_t^{1-\alpha}$, we only proved an $O(k^{1+\alpha})$ convergence for $0 < \alpha < 1$ in the case of non-uniform time meshes, where k denotes the maximum time step. A fully discrete scheme that combined finite elements in space with Crank-Nicolson in time was proposed, and the existence and uniqueness of the solution of the fully discrete scheme was proved. We introduced another numerical scheme based on $L1$ approximation in time and finite elements in space, and we performed the error analysis for the fully discrete scheme. We got results better than the first method; we got an order of $O(k^2)$ convergence rate in the case of non-uniform time meshes.

In comparison to the previous work regarding the convergence rate, we find that our results are better than the work done by Le et al. [7]. In their work, they proved an $O(k^\alpha)$ order of convergence. However, in our numerical methods, we got $O(k\alpha{+}1)$ using the Crank-Nicolson method and $O(k2)$ using the $L1$ approximation scheme.

Future work

In the future we plan to investigate the numerical solution of the time-fractional Fokker-Planck equation in the case of non-smooth initial data.

Acknowledgement. I would like to thank the reviewer for their effort to review my paper.

Conflict of interest. The paper is written by a single author so there is no conflict of interest.

Authors' Contributions. It is a single-author paper. So, full credit goes to the author.

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