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The Maximum Degree, the Independence Number and Some Hamiltonian Properties of a Graph

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Abstract. In this note, we present sufficient conditions involving the maximum degree and the independence number for some Hamiltonian properties of a graph.

Keywords: The maximum degree, the independence number, Hamiltonian graph, traceable graph

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1. Introduction

We consider only finite undirected graphs without loops or multiple edges. Notation and terminology not defined here follow those in [1]. Let G = (V(G), E(G)) be a graph with n vertices and e edges. We use δ and Δ to denote the minimum and maximum degrees of a graph G, respectively. A subset S of V(G) in G is independent if there is no edge between any pair of distinct vertices in S. A maximum independent set in a graph G is an independent set of maximum cardinality. The independence number of a graph G, denoted $\alpha(G)$, is the size of a maximum independent set in G. For disjoint vertex subsets X and Y of V(G), we define E(X, Y) as $\{e : e = xy \in E, x \in X, y \in Y\}$. We use $K_{r, s}$ to denote a complete bipartite graph in which the two partition sets have cardinalities of r and s, respectively. A cycle C in a graph G is called a Hamilton cycle of G if C contains all the vertices of G. A graph G is called Hamiltonian if G has a Hamilton cycle. A path P in a graph G is called a Hamilton path of G if P contains all the vertices of G. A graph G is called traceable if G has a Hamilton path. In this note, we present sufficient conditions involving the maximum degree and the independence number for Hamiltonian graphs and traceable graphs. The main results are as follows.

Theorem 1.1. Let G be a k-connected graph ($k \geq 2$) with $n \geq 3$ vertices, e edges, and maximum degree Δ. If $e \geq (n - k - 1)\,\Delta$, then G is Hamiltonian or G is $K_{k, k+1}$.

Theorem 1.2. Let G be a k-connected graph ($k \geq 1$) with $n \geq 9$ vertices, e edges, and maximum degree Δ. If $e \geq (n - k - 2)\,\Delta$, then G is traceable or G is $K_{k, k+2}$.

2. Lemmas

We will use the following results as lemmas in our proofs of Theorem 1 and Theorem 2. Lemma 2.1 and Lemma 2.2 below are from [2].

Lemma 2.1 [2]. Let G be a k-connected graph of order $n \geq 3$. If $\alpha \leq k$, then G is Hamiltonian.

Lemma 2.2 [2]. Let G be a k-connected graph of order n. If $\alpha \leq k + 1$, then G is traceable.

Lemma 2.3 below is from [4].

Lemma 2.3 [4]. Let G be a balanced bipartite graph of order 2n with bipartition (A, B). If $d(x) + d(y) \geq n + 1$ for any $x \in A$ and any $y \in B$ with $xy \notin E$, then G is Hamiltonian.

Lemma 2.4 below is from [3].

Lemma 2.4 [3]. Let G be a 2-connected bipartite graph with bipartition (A, B), where $|A| \geq |B|$. If each vertex in A has degree at least s and each vertex in B has degree at least t, then G contains a cycle of length at least $2\min(|B|, s + t - 1, 2s - 2)$.

3. Proofs

Proof of Theorem 1.1. Let G be a k-connected graph ($k \geq 2$) with $n \geq 3$ vertices, e edges, and the maximum degree Δ satisfying the conditions in Theorem 1.1. Suppose G is not Hamiltonian. Then we have that $n \geq 2\delta + 1 \geq 2k + 1$ otherwise $\delta \geq k \geq n/2$ and G is Hamiltonian. In addition, Lemma 2.1 implies that $\alpha \geq k + 1$. Let I be any maximum independent set in G. Then $|I| = \alpha < n$. Thus

$$\sum_{u \in I} d(u) = |E(I, V - I)| \leq \sum_{v \in V - I} d(v).$$

Since $\sum_{u \in I} d(u) + \sum_{v \in V - I} d(v) = 2e$, we have that

$$\sum_{u \in I} d(u) \leq e \leq \sum_{v \in V - I} d(v).$$

Notice that $d(v) \leq \Delta$ for each vertex $v \in V - I$. We have that

$$(n - k - 1)\,\Delta \leq e \leq \sum_{v \in V - I} d(v) \leq (n - \alpha)\,\Delta \leq (n - k - 1)\,\Delta.$$

Then

$$(n - k - 1)\,\Delta = e = \sum_{v \in V - I} d(v) = (n - \alpha)\,\Delta = (n - k - 1)\,\Delta.$$

Thus $d(v) = \Delta$ for each $v \in V - I$, $|I| = \alpha = k + 1$, and $\sum_{v \in V - I} d(v) = e$ which implies that $\sum_{u \in I} d(u) = e$ and G is a bipartite graph with partition sets I and V - I.
Since $n \geq 2k + 1$, we have that $|V - I| \geq k$. If $|V - I| \geq k + 2$, then V - I is an independent set with size at least $k + 2$, contradicting to the fact that I is a maximum independent set in G with size $k + 1$. Thus

$|V - I| = k + 1$ or $|V - I| = k$. If $|V - I| = k + 1$, then Lemma 2.3 implies that G is Hamiltonian, a contradiction. If $|V - I| = k$, then G is $K_{k, k + 1}$.

This completes the proof of Theorem 1.1.

Proof of Theorem 1.2. Let G be a k-connected graph ($k \geq 1$) with $n \geq 9$ vertices, e edges, and the maximum degree Δ satisfying the conditions in Theorem 1.2. Suppose G is not traceable. Then we have that $n \geq 2\delta + 2 \geq 2k + 2$ otherwise $\delta \geq k \geq (n-1)/2$ and G is traceable. In addition, Lemma 2.2 implies that $\alpha \geq k + 2$. Let I be any maximum independent set in G. Then $|I| = \alpha < n$. Thus

$$\sum_{u \in I} d(u) = |E(I, V - I)| \leq \sum_{v \in V - I} d(v).$$

Since $\sum_{u \in I} d(u) + \sum_{v \in V - I} d(v) = 2e$, we have that

$$\sum_{u \in I} d(u) \leq e \leq \sum_{v \in V - I} d(v).$$

Notice that $d(v) \leq \Delta$ for each vertex $v \in V - I$. We have that

$$(n - k - 2)\, \Delta \leq e \leq \sum_{v \in V - I} d(v) \leq (n - \alpha)\, \Delta \leq (n - k - 2)\, \Delta.$$

Then

$$(n - k - 2)\, \Delta = e = \sum_{v \in V - I} d(v) = (n - \alpha)\, \Delta = (n - k - 2)\, \Delta.$$

Thus $d(v) = \Delta$ for each $v \in V - I$, $|I| = \alpha = k + 2$, and $\sum_{v \in V - I} d(v) = e$ which implies that $\sum_{u \in I} d(u) = e$ and G is a bipartite graph with partition sets I and V - I. Since $n \geq 2k + 2$, we have that $|V - I| \geq k$. If $|V - I| \geq k + 3$, then V - I is an independent set with size at least $k + 3$, contradicting to the fact that I is a maximum independent set in G with size $k + 2$.

Thus $|V - I| = k + 2$, $|V - I| = k + 1$, or $|V - I| = k$. If $|V - I| = k + 2$, then $k \geq 3$ and Lemma 2.3 implies that G is Hamiltonian and thereby G is traceable, a contradiction. If $|V - I| = k + 1$, then $k \geq 3$ and Lemma 2.4 implies that G has a cycle of length at least $(n - 1)$ and thereby G is traceable, a contradiction. If $|V - I| = k$, then G is $K_{k,\, k+2}$.

This completes the proof of Theorem 1.2.

4. Conclusion

In this note, we present new sufficient conditions that involve the maximum degree and the independence number for Hamiltonian and traceable graphs.

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Conflict of interest. The paper is written by a single author so there is no conflict of interest.

Authors' Contributions. It is a single-author paper. So, full credit goes to the author.

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