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Uphill Kepler Banhatti and Modified Uphill Kepler Banhatti Indices of Graphs

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Abstract. In this study, we introduce the uphill Kepler Banhatti and modified uphill Kepler Banhatti indices and their corresponding exponentials of a graph. Furthermore, we compute these indices for wheel graphs. Also we obtain some properties of uphill Kepler Banhatti index.

Keywords: uphill Kepler Banhatti index, modified uphill Kepler Banhatti index, graphs.

AMS Mathematics Subject Classification (2010): 05C10, 05C69

1. Introduction

In this paper, G denotes a finite, simple, connected graph, $V(G)$ and $E(G)$ denote the vertex set and edge set of G. The degree d_u of a vertex u is the number of vertices adjacent to u. Graph indices have their applications in various disciplines of Science and Technology.

A u-v path P in G is a sequence of vertices in G, starting with u and ending at v, such that consecutive vertices in P are adjacent, and no vertex is repeated. A path $\pi = v_1, v_2, \ldots v_{k+1}$ in G is a uphill path if for every i, $1 \leq i \leq k$, $d_G(v_i) \leq d_G(v_{i+1})$.

A vertex v is uphill dominating a vertex u if there exists an uphill path originated from u to v. The uphill neighborhood of a vertex v is denoted by $N_{up}(v)$ and defined as: $N_{up}(v) = \{u: v \text{ uphill dominates } u\}$. The uphill degree $d_{up}(v)$ of a vertex v is the number of uphill neighbors of v, see [1, 2].

The modified first uphill Zagreb index [1] of a graph is defined as

$$UPM_1^*(G) = \sum_{uv \in E(G)} \left(d_{up}(u) + d_{up}(v)\right).$$

Ref. [1] was soon followed by a series of publications [3, 4].
The uphill Sombor index was introduced in [4] and it is defined as

$$USO(G) = \sum_{uv \in E(G)} \sqrt{d_{up}(u)^2 + d_{up}(v)^2}.$$

The reciprocal uphill product connectivity index of a graph G *is* defined as

$$RUP(G)=\sum_{uv\in E(G)}\sqrt{d_{up}(u)d_{up}(v)}.$$

The Kepler Banhatti index was introduced by Kulli in [5] and it is defined as

$$KB(G)=\sum_{uv\in E(G)}\left((d_u+d_v)+\sqrt{d_u^2+d_v^2}\right).$$

Motivated by the definition of Kepler Banhatti index, we introduce the uphill Kepler Banhatti index of a graph and it is defined as

$$UPKB(G)=\sum_{uv\in E(G)}\left(d_{up}(u)+d_{up}(v)+\sqrt{d_{up}(u)^2+d_{up}(v)^2}\right)$$

Considering the uphill Kepler Banhatti index, we introduce the uphill Kepler Banhatti exponential of a graph G and defined it as

$$UPKB(G,x)=\sum_{uv\in E(G)}x^{d_{up}(u)+d_{up}(v)+\sqrt{d_{up}(u)^2+d_{up}(v)^2}}.$$

We define the modified uphill Kepler Banhatti index of a graph G as

$$^{m}UPKB(G)=\sum_{uv\in E(G)}\frac{1}{d_{up}(u)+d_{up}(v)+\sqrt{d_{up}(u)^2+d_{up}(v)^2}}.$$

Considering the modified uphill Kepler Banhatti index, we introduce the modified uphill Kepler Banhatti exponential of a graph G and defined it as

$$^{m}UPKB(G,x)=\sum_{uv\in E(G)}x^{\frac{1}{d_{up}(u)+d_{up}(v)+\sqrt{d_{up}(u)^2+d_{up}(v)^2}}}.$$

Recently, some topological indices were studied in [6, 7, 8].
In this paper, the uphill Kepler Banhatti index, modified uphill Kepler Banhatti index and their corresponding exponentials of wheel graphs are computed.

2. Mathematical properties

Theorem 1. Let G be a simple connected graph. Then

$$UPKB(G)\geq\left(1+\frac{1}{\sqrt{2}}\right)UPM_1^*(G)$$

with equality if G is regular.

Proof: It is known that

$$\sqrt{\frac{d_{up}(u)^2+d_{up}(v)^2}{2}}\geq\frac{\left(d_{up}(u)+d_{up}(v)\right)}{2}$$

thus

$$\left(d_{up}(u)+d_{up}(v)\right)+\sqrt{d_{up}(u)^2+d_{up}(v)^2}\geq\left(d_{up}(u)+d_{up}(v)\right)+\frac{1}{\sqrt{2}}\left(d_{up}(u)+d_{up}(v)\right).$$

Hence

$$\sum_{uv\in E(G)}[\left(d_{up}(u)+d_{up}(v)\right)+\sqrt{d_{up}(u)^2+d_{up}(v)^2}]\geq\left(1+\frac{1}{\sqrt{2}}\right)\sum_{uv\in E(G)}\left(d_{up}(u)+d_{up}(v)\right).$$

Thus $$UPKB(G) \geq \left(1+\frac{1}{\sqrt{2}}\right) UPM_1^*(G)$$

with equality if G is regular.

Theorem 2. Let G be a simple connected graph. Then
$$UPKB(G) \leq \left(1+\sqrt{2}\right) UPM_1^*(G) - \sqrt{2}RUP(G).$$
Proof: It is known that for $1 \leq x \leq y$,
$$f(x,y) = \left(x+y-\sqrt{xy}\right) - \sqrt{\frac{x^2+y^2}{2}}$$
is decreasing for each y. Thus $f(x,y) \geq f(y,y) = 0$. Hence
$$x+y-\sqrt{xy} \geq \sqrt{\frac{x^2+y^2}{2}}$$
or $$\sqrt{\frac{x^2+y^2}{2}} \leq x+y-\sqrt{xy}.$$
Put $x = d_{up}(u)$ and $y = d_{up}(v)$, we get
$$\sqrt{\frac{d_{up}(u)^2+d_{up}(v)^2}{2}} \leq \left(d_{up}(u)+d_{up}(v)\right) - \sqrt{d_{up}(u)d_{up}(v)}$$
$$\sqrt{d_{up}(u)^2+d_{up}(v)^2} \leq \sqrt{2}[\left(d_{up}(u)+d_{up}(v)\right) - \sqrt{d_{up}(u)d_{up}(v)}]$$
which implies
$$\left(d_{up}(u)+d_{up}(v)\right) + \sqrt{d_{up}(u)^2+d_{up}(v)^2}$$
$$\leq \left(d_{up}(u)+d_{up}(v)\right) + \sqrt{2}[\left(d_{up}(u)+d_{up}(v)\right) - \sqrt{d_{up}(u)d_{up}(v)}]$$
$$\sum_{uv\in E(G)} [d_{up}(u)+d_{up}(v)+\sqrt{d_{up}(u)^2+d_{up}(v)^2}]$$
$$\leq \left(1+\sqrt{2}\right)\sum_{uv\in E(G)}\left(d_{up}(u)+d_{up}(v)\right) - \sqrt{2}\sum_{uv\in E(G)}\sqrt{d_{up}(u)d_{up}(v)}$$
Thus $$UPKB(G) \leq \left(1+\sqrt{2}\right) UPM_1^*(G) - \sqrt{2}RUP(G).$$

Theorem 3. Let G be a simple connected graph. Then
$$UPKB(G) < 2UPM_1^*(G).$$
Proof: It is known that for $1 \leq x \leq y$,
$$\sqrt{x^2+y^2} < x+y$$
$$(x+y)+\sqrt{x^2+y^2} < 2(x+y).$$
Setting $x = d_d(u)$ and $y = d_d(v)$, we get
$$\left(d_{up}(u)+d_{up}(v)\right) + \sqrt{d_{up}(u)^2+d_{up}(v)^2} < 2\left(d_{up}(u)+d_{up}(v)\right).$$

Thus $\sum_{uv\in E(G)}[\left(d_{up}(u)+d_{up}(v)\right)+\sqrt{d_{up}(u)^2+d_{up}(v)^2}]<2\sum_{uv\in E(G)}\left(d_{up}(u)+d_{up}(v)\right).$

Hence $$UPKB(G)<2UPM_1^*(G).$$

Theorem 4. Let G be a simple connected graph. Then

$$UPKB(G)=UPM_1^*(G)+USO(G).$$

Proof: We have

$$\sum_{uv\in E(G)}[\left(d_{up}(u)+d_{up}(v)\right)+\sqrt{d_{up}(u)^2+d_{up}(v)^2}]$$

$$=\sum_{uv\in E(G)}\left(d_{up}(u)+d_{up}(v)\right)+\sum_{uv\in E(G)}\sqrt{d_{up}(u)^2+d_{up}(v)^2}$$

Hence $$UPKB(G)=UPM_1^*(G)+USO(G).$$

3. Results for wheel graphs

Let W_n be a wheel with n+1vertices and 2n edges, n□4. Then there are two types of edges based on the uphill degree of end vertices of each edge as follows:

$E_1 = \{uv \in E(W_n) \mid d_{up}(u) = 0, d_{up}(v) = n\}$, $\quad |E_1| = n.$

$E_2 = \{uv \in E(W_n) \mid d_{up}(u) = d_{up}(v) = n\}$, $\quad |E_2| = n.$

Theorem 5. Let W_n be a wheel with n+1vertices and 2n edges, $n\geq 4$. Then

$$UPKB(W_n)=\left(4+\sqrt{2}\right)n^2.$$

Proof. We deduce

$$UPKB(W_n)=\sum_{uv\in E(W_n)}\left(d_{up}(u)+d_{up}(v)+\sqrt{d_{up}(u)^2+d_{up}(v)^2}\right)$$

$$=n\left(0+n+\sqrt{0^2+n^2}\right)+n\left(n+n+\sqrt{n^2+n^2}\right)=\left(4+\sqrt{2}\right)n^2.$$

Theorem 6. Let W_n be a wheel with n+1vertices and 2n edges, $n\geq 4$. Then

$$UPKB(W_n,x)=nx^{2n}+nx^{\left(2+\sqrt{2}\right)n}.$$

Proof. We deduce

$$UPKB(W_n,x)=\sum_{uv\in E(W_n)}x^{d_{up}(u)+d_{up}(v)+\sqrt{d_{up}(u)^2+d_{up}(v)^2}}$$

$$=nx^{0+n+\sqrt{0^2+n^2}}+nx^{n+n+\sqrt{n^2+n^2}}=nx^{2n}+nx^{\left(2+\sqrt{2}\right)n}.$$

$$=\left(4+\sqrt{2}\right)n^2.$$

Theorem 7. Let W_n be a wheel with n+1vertices and 2n edges, $n\geq 4$. Then

$$^{m}UPKB(W_n)=\frac{1}{2}+\frac{1}{2+\sqrt{2}}.$$

Proof. We deduce

$$ {}^{m}UPKB(W_n) = \sum_{uv \in E(W_n)} \frac{1}{d_{up}(u) + d_{up}(v) + \sqrt{d_{up}(u)^2 + d_{up}(v)^2}} $$
$$ = \frac{n}{0 + n + \sqrt{0^2 + n^2}} + \frac{n}{n + n + \sqrt{n^2 + n^2}} = \frac{1}{2} + \frac{1}{2 + \sqrt{2}}. $$

Theorem 8. Let W_n be a wheel with n+1vertices and 2n edges, $n \geq 4$. Then

$$ {}^{m}UPKB(W_n, x) = nx^{\frac{1}{2n}} + nx^{\frac{1}{(2+\sqrt{2})n}}. $$

Proof. We deduce

$$ {}^{m}UPKB(W_n, x) = \sum_{uv \in E(W_n)} x^{\frac{1}{d_{up}(u) + d_{up}(v) + \sqrt{d_{up}(u)^2 + d_{up}(v)^2}}} $$
$$ = nx^{\frac{1}{0+n+\sqrt{0^2+n^2}}} + nx^{\frac{1}{n+n+\sqrt{n^2+n^2}}} = nx^{\frac{1}{2n}} + nx^{\frac{1}{(2+\sqrt{2})n}}. $$

4. Conclusion

In this paper, the uphill Kepler Banhatti, modified uphill Kepler Banhatti indices and their corresponding exponentials of wheel graphs are computed. Also we have established some properties of uphill Kepler Banhatti index.

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