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Short Communication

An Upper Bound for the Independence Number and Some Hamiltonian Properties of a Graph

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Abstract. In this note, we present an upper bound for the independence number of a graph. Using that upper bound, we obtain sufficient conditions for some Hamiltonian properties of a graph.

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1. Introduction

We consider only finite undirected graphs without loops or multiple edges. Notation and terminology not defined here follow those in [1]. Let $G = (V(G), E(G))$ be a graph with n vertices and e edges. We use δ and Δ to denote the minimum and maximum degrees of a graph G, respectively. A subset S of $V(G)$ in G is independent if there is no edge between any pair of distinct vertices in S. A maximum independent set in a graph G is an independent set of maximum cardinality. The independence number of a graph G, denoted $\alpha(G)$, is the size of a maximum independent set in G. For two graphs H_1 and H_2, we define $H_1 \leq H_2$ as $V(H_1) = V(H_2)$ and $E(H_1) \subseteq E(H_2)$. The join of two disjoint graphs H_1 and H_2 is denoted by $H_1 \vee H_2$. We use $K_{p,q}$ to denote a complete bipartite graph in which the two partition sets have cardinalities of p and q, respectively. For disjoint vertex subsets X and Y of $V(G)$, we define $E(X, Y)$ as $\{e : e = xy \in E, x \in X, y \in Y\}$. A cycle C in a graph G is called a Hamilton cycle of G if C contains all the vertices of G. A graph G is called Hamiltonian if G has a Hamilton cycle. A path P in a graph G is called a Hamilton path of G if P contains all the vertices of G. A graph G is called traceable if G has a Hamilton path. In this note, we first present an upper bound for the independence number of a graph. Using that upper bound, we obtain sufficient conditions for Hamiltonian graphs and traceable graphs. The main results of this note are as follows.

Theorem 1.1. Let G be a graph with n vertices, e edges, maximum degree Δ, and minimum degree $\delta \geq 1$. Then

$$\alpha \leq \frac{n - \Delta + \sqrt{(n - \Delta)^2 + 4(n\Delta - 2e)}}{2}$$

with equality if and only if G is $K_{\alpha}^{c} \vee H$, where $H = G[V - I]$, I is a maximum independent set in G with $|I| = \alpha$ such that $d(u) = \delta = (n - \alpha)$ for each $u \in I$ and $d(v) = \Delta$ for each $v \in V - I$.

Theorem 1.2. Let G be a k-connected ($k \geq 2$) graph with $n \geq 3$ vertices and e edges. If

$$\frac{n - \Delta + \sqrt{(n - \Delta)^2 + 4(n\Delta - 2e)}}{2} \leq k + 1,$$

then G is Hamiltonian or $G \in \{H : K_{k, k+1} \leq H \leq K_{k+1}^{c} \vee H$ and $d_H(v) = \Delta$ if $v \in V$ and $d_H(v) \neq \delta = k\}$.

Theorem 1.3. Let G be a k-connected ($k \geq 1$) graph with n vertices and e edges. If

$$\frac{n - \Delta + \sqrt{(n - \Delta)^2 + 4(n\Delta - 2e)}}{2} \leq k + 2,$$

then G is traceable or $G \in \{H : K_{k, k+2} \leq H \leq K_{k+2}^{c} \vee H$ and $d_H(v) = \Delta$ if $v \in V$ and $d_H(v) \neq \delta = k\}$.

2. Lemmas

We will use the following results as lemmas in our proofs of Theorem 1.2 and Theorem 1.3. Lemma 2.1 and Lemma 2.2 below are from [2].

Lemma 2.1 [2]. Let G be a k-connected graph of order $n \geq 3$. If $\alpha \leq k$, then G is Hamiltonian.

Lemma 2.2 [2]. Let G be a k-connected graph of order n. If $\alpha \leq k + 1$, then G is traceable.

3. Proofs

Proof of Theorem 1.1. Let G be a graph with n vertices, e edges, maximum degree Δ, and minimum degree $\delta \geq 1$. Let I be any maximum independent set in G. Then $|I| = \alpha < n$. Notice that $d(u) \leq n - \alpha$ for each u ⍰ I and $d(v) \leq \Delta$ for each v ⍰ $V - I$. We have that

$$2e = \sum_{u \in I} d(u) + \sum_{v \in V - I} d(v) \leq \alpha(n - \alpha) + (n - \alpha)\Delta.$$

Solving the quadratic equation for α, we have that

$$\alpha \leq \frac{n - \Delta + \sqrt{(n - \Delta)^2 + 4(n\Delta - 2e)}}{2}.$$

Suppose that

$$\alpha = \frac{n - \Delta + \sqrt{(n - \Delta)^2 + 4(n\Delta - 2e)}}{2}.$$

Then

$$2e = \sum_{u \in I} d(u) + \sum_{v \in V - I} d(v) \leq \alpha(n - \alpha) + (n - \alpha)\Delta.$$

Thus $d(u) = n - \alpha$ for each $u \in I$ and $d(v) = \Delta$ for each $v \in V - I$. Therefore G is $K_\alpha^c \vee H$, where $H = G[V - I]$, I is a maximum independent set in G with $|I| = \alpha$ such that $d(u) = \delta = (n - \alpha)$ for each $u \in I$ and $d(v) = \Delta$ for each $v \in V - I$.

If G is $K_\alpha^c \vee H$, where $H = G[V - I]$, I is a maximum independent set in G with $|I| = \alpha$ such that $d(u) = \delta = (n - \alpha)$ for each $u \in I$ and $d(v) = \Delta$ for each $v \in V - I$. Then

$$2e = \sum_{u \in I} d(u) + \sum_{v \in V - I} d(v) = \alpha(n - \alpha) + (n - \alpha)\Delta.$$

Thus

$$(n - \Delta)^2 + 4(n\Delta - 2e) = (n - \Delta)^2 + 4(n\Delta - \alpha(n - \alpha) - (n - \alpha)\Delta)$$
$$= (\Delta - n)^2 + 4\alpha(\Delta - n) + 4\alpha^2 = (2\alpha + \Delta - n)^2.$$

Let u be a vertex in I. Then $\Delta \geq d(u) = (n - \alpha)$. Thus we have that $(2\alpha + \Delta - n) > 0$. Therefore

$$\frac{n - \Delta + \sqrt{(n - \Delta)^2 + 4(n\Delta - 2e)}}{2}$$

$$= \frac{n - \Delta + \sqrt{(2\alpha + \Delta - n)^2}}{2}$$

$$= \frac{n - \Delta + (2\alpha + \Delta - n)}{2} = \alpha.$$

This completes the proof of Theorem 1.1.

Proof of Theorem 1.2. Let G be a k-connected ($k \geq 2$) graph with $n \geq 3$ vertices and e edges satisfying the conditions in Theorem 1.2. Suppose G is not Hamiltonian. Then Lemma 2.1 implies that $\alpha \geq k + 1$. Also, we have that $n \geq 2\delta + 1 \geq 2k + 1$ otherwise $\delta \geq k \geq n/2$ and G is Hamiltonian. Let I be any maximum independent set in G. Then $|I| = \alpha$. From the proof of Theorem 1.1, we have that

$$k + 1 \leq |I| \leq \alpha \leq \frac{n - \Delta + \sqrt{(n - \Delta)^2 + 4(n\Delta - 2e)}}{2} \leq k + 1.$$

Thus

$$k + 1 = |I| = \alpha = \frac{n - \Delta + \sqrt{(n - \Delta)^2 + 4(n\Delta - 2e)}}{2}.$$

From the proof of Theorem 1.1 again, we have that $d(u) = \delta = (n - \alpha) = (n - k - 1)$ for each $u \in I$ and $d(v) = \Delta$ for each $v \in V - I$. Hence G is $K_{k+1}^c \vee H$, where $H = G[V - I]$, I is a maximum independent such that in G with $|I| = k + 1$ such that $d(u) = \delta = (n - k - 1)$ for each $u \in I$ and $d(v) = \Delta$ for each $v \in V - I$.

If $n - k - 1 \geq k + 1$, then $\delta = n - k - 1 \geq (n - k - 1 + k + 1)/2 = n/2$ and G is Hamiltonian, a contradiction. If $n - k - 1 \leq k$, then $n \leq 2k + 1$. Thus $n = 2k + 1$ and $G \in \{H : K_{k, k+1} \leq H \leq K_{k+1}^c \vee H$ and $d_H(v) = \Delta$ if $v \in V$ and $d_H(v) \neq \delta = k\}$.

This completes the proof of Theorem 1.2.

Proof of Theorem 1.3. Let G be a k-connected graph ($k \geq 1$) with n vertices and e edges satisfying the conditions in Theorem 1.3. Suppose G is not traceable. Then Lemma 2.2 implies that $\alpha \geq k + 2$. Also, we have that $n \geq 2\delta + 2 \geq 2k + 2$ *otherwise* $\delta \geq k \geq (n - 1)/2$ and G is traceable. Let I be any maximum independent set in G. Then $|I| = \alpha$. From the proof of Theorem 1.1, we have that

$$\boldsymbol{k+2 \ \leq |I| \ \leq \ \alpha \leq \frac{n - \ \Delta + \sqrt{(n - \ \Delta)^2 \ + 4(n\Delta - 2e)}}{2} \leq k+2}.$$

Thus

$$\boldsymbol{k+2 = |I| = \ \alpha = \frac{n - \ \Delta + \sqrt{(n - \ \Delta)^2 \ + 4(n\Delta - 2e)}}{2}}.$$

From the proof of Theorem 1.1 again, we have that $d(u) = \delta = (n - \alpha) = (n - k - 2)$ for each $u \in \boldsymbol{I}$ and $d(v) = \Delta$ for each $v \in V - \boldsymbol{I}$. Hence G is $K^c_{k+2} \vee H$, where $H = G[V - I]$, I is a maximum independent such that in G with $|I| = k + 2$ such that $d(u) = \delta = (n - k - 2)$ for each $u \in I$ and $d(v) = \Delta$ for each $v \in V - I$.

If $n - k - 2 \geq k + 1$, then $\delta = n - k - 2 \geq (n - k - 2 + k + 1)/2 = (n - 1)/2$ and G is traceable, a contradiction. If $n - k - 2 \leq k$, then $n \leq 2k + 2$. Thus $n = 2k + 2$ and $G \in \{H : K_{k,\, k+2} \leq H \leq K^c_{k+2} \vee H$ and $d_H(v) = \Delta$ if $v \in V$ and $d_H(v) \neq \delta = k\}$.

This completes the proof of Theorem 1.3.

4. Conclusion

In this note, we present a new upper bound for the independence number of a graph. Using that upper bound, we obtain new sufficient conditions involving the maximum degree for Hamiltonian and traceable graphs.

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