

Dominating b-Colour Transversal Sets in Graphs

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Received 28 January 2026; accepted 14 March 2026

Abstract. In the present paper, we have defined a new set in a graph, called a dominating b-colour transversal set, which is both a dominating set and a b-colour transversal in some b-chromatic partition of the graph's vertices. Here, a b-colour transversal is a set that contains at least one b-vertex of each colour class of some b-chromatic partition of the graph. The cardinality of the minimum dominating b-colour transversal set is called the dominating b-colour transversal number of the graph. We find a relation of this number with the domination number and the b-chromatic number of a graph. In the process, we find an upper bound of this number in terms of domination number and b-chromatic number. Furthermore, we obtain a characterisation of the minimal dominating b-colour transversal set of a graph.

Keywords: dominating set; b-chromatic number; b-colour transversal set.

AMS Mathematics Subject Classification (2010): 05C25, 05C15, 05D15

1. Introduction

If $G = (V, E)$ is a graph, then a proper k -colouring of the vertices of G is a mapping $c: V \rightarrow \{1, 2, 3, \dots, k\}$ such that $c(u) \neq c(v)$, whenever u and v are adjacent. This colouring partitions V into independent subsets V_1, V_2, \dots, V_k . Each subset is called a colour class. If integer k is the minimum such that G admits a proper k -colouring, then k is called the chromatic number of G , and it is denoted by $\chi(G)$. Note that proper colouring of vertices with $\chi(G)$ colours is called chromatic colouring. A b-colouring of a graph G is a proper colouring of G in which each colour class contains a vertex, called a b-vertex, that is adjacent to at least one vertex of each of the other colour classes. This concept of b-colouring was introduced by Irvin and Manlove in 1999 [4]. The b-chromatic number of G , denoted by $\phi(G)$, is the largest integer k such that G admits a b-colouring using k colours.

A subset S of V is said to be a dominating set of G if for each $v \in V$ either $v \in S$ or v is adjacent to some vertex in S . The dominating set with minimum cardinality is called γ -Set of G and the cardinality of S is called domination number of the graph G , denoted by $\gamma(G)$ or just by γ .

In [8], Thakkar and Kothiya combined two concepts from graph theory, viz., total domination and colouring, and defined a total dominating colour transversal set in graphs, thereby obtaining many interesting results. Motivated by their work, we introduced a

concept called the dominating b -colour transversal set in graphs, which is an amalgam of two concepts, viz., domination and b -colouring in graphs.

We make the following convention: Throughout the article, we consider simple, finite, undirected graphs, unless otherwise stated. We have followed the references [1], [3], and [7] for the fundamentals of graph theory, notation, and terminology. We start with definitions.

2. Definitions

Definition 1.1. [4] The m -degree of a graph G , denoted by $m(G)$, is the largest integer m such that G has m vertices of degree at least $m - 1$.

Definition 1.2. [4] Let $G = (V, E)$ be a graph and $\phi(G) = k$. Then vertex V can be partitioned into colour classes V_1, V_2, \dots, V_k . The set $\{V_1, V_2, \dots, V_k\}$ is χ^b -Partition or b -chromatic Partition of G .

Definition 1.3. Let $G = (V, E)$ be a graph with $\phi(G) = k$ and χ^b -Partition $\{V_1, V_2, \dots, V_k\}$. Then $S \subset V$ is called a b -colour transversal set if S contains at least one b -vertex of each colour class V_i .

Definition 1.4. Let $G = (V, E)$ be a graph. A dominating set S of G which is also a b -colour transversal of some χ^b -Partition of G is called a dominating b -colour transversal set of G , and it is denoted by dbt -set.

Definition 1.5. Let $G = (V, E)$ be a graph. Then a dominating b -colour transversal set with minimum cardinality is called a minimum dominating b -colour transversal set of G , and it is noted by γ_{dbt} -set. The cardinality of minimum dominating b -colour transversal set of G is called dominating b -colour transversal number of G and it is denoted by $\gamma_{dbt}(G)$.

Definition 1.6. Let $G = (V, E)$ be a graph and S be a dominating colour b -transversal set of G . Then S is called a minimal dominating b -colour transversal of G if no proper subset of S is a dominating b -colour transversal set of G .

3. Main results

Proposition 3.1. [4] For any graph G , $\chi(G) \leq \phi(G) \leq m(G)$.

Proposition 3.2. For any graph G , $\gamma(G), \phi(G) \leq \gamma_{dbt}(G) \leq \gamma(G) + \phi(G)$.

Proof: Obvious.

Remark 3.3. Upper bound mentioned above is tight. See the following graph G .

Clearly, $m(G) = 4$. So $\phi(G) = 4$ with b vertices u_1, u_2, u_3 and u_4 . Note that there are four pendant vertices u_6, u_7, u_{11} and u_{12} . So at least four vertices are required to dominate them. So $\{u_5, u_8, u_9, u_{10}\}$ is a minimum dominating set and so $\gamma(G) = 4$.

No vertices other than u_1, u_2, u_3 and u_4 can be a b -vertex as their degree is less than 3. So any b -colour transversal with four colours must contain these b -vertices and to

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dominate all the pendant vertices $\{u_5, u_8, u_9, u_{10}\}$ is required. In fact, as noted above this set $\{u_5, u_8, u_9, u_{10}\}$ is a dominating set. So $\gamma_{abt}(G) = 8 = \gamma(G) + \emptyset(G)$ with γ_{abt} - set $\{u_1, u_2, u_3, u_4, u_5, u_8, u_9, u_{10}\}$.

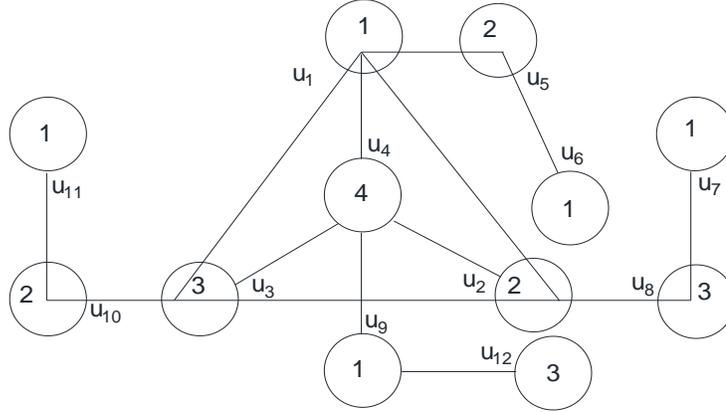


Figure 1: Graph G

Theorem 3.4. Let G be a graph. Then $\gamma_{abt}(G) = \gamma(G) + \emptyset(G)$ if and only if for every γ - set D of G, the following conditions hold:

- 1). $D \cap \{v : v \text{ is a } b\text{-vertex of some } \chi^b\text{-Partition of } G\} = \emptyset$.
- 2). D is contained in some γ_{abt} - set S of G.

Proof: Let $\gamma_{abt}(G) = \gamma(G) + \emptyset(G)$ and D be a γ - set of G. Assume, if possible, that condition 1) does not hold. Then D contains a b - vertex, say v, of some χ^b - Partition $\{V_1, V_2, V_3, \dots, V_{\emptyset(G)}\}$ of G. Let $v \in V_i$, for some i. Adding a b - vertex, from each of remaining colour $\emptyset(G) - 1$ classes, to D, yields a dominating b - colour transversal set of G with cardinality $\gamma(G) + \emptyset(G) - 1$, which is not possible. So condition 1) holds. Now add, to D, one b - vertex from each of the colour class of some χ^b - Partition of G, we get a γ_{abt} - set S of G of cardinality $\gamma(G) + \emptyset(G)$. Hence D is contained in some γ_{abt} - set S of G. Conversely, assume conditions 1) and 2). Then $\emptyset(G) \leq |S \setminus D| \leq |S| - |D| = \gamma_{abt}(G) - \gamma(G)$. So $\gamma_{abt}(G) \geq \gamma(G) + \emptyset(G)$. But we know that $\gamma_{abt}(G) \leq \gamma(G) + \emptyset(G)$. So $\gamma_{abt}(G) = \gamma(G) + \emptyset(G)$.

Theorem 3.5. Let any graph G with n vertices and u be a vertex of G. Then $\{u\}$ is a colour class for every χ^b - Partition if and only if $\deg(u) = n - 1$.

Proof: Let G be a graph with n vertices and u be a vertex of G. Suppose $\{u\}$ is a colour class for every χ^b - Partition. Then u is a b - vertex for every χ^b - Partition. If possible, assume that $\deg(u) \leq n - 2$. Then u is not adjacent to at least one vertex, say v. If $\{v\}$ is a colour class then u being a b - vertex it must be adjacent to v. So $\{v\}$ is not a colour class. Also note that v cannot be b vertex for any χ^b - Partition as otherwise v need to be adjacent to u. So we can generate a colour class $\{u, v\}$ without reducing the number of colour

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classes, which is contradiction to the fact that $\{u\}$ is a colour class for every χ^b - Partition. Hence $\deg(u) = n - 1$. Converse is obvious.

Remark 3.6. Consider the cycle graph C_4 . Note that $\emptyset(C_4) = 2$. It is uniquely b - colourable graph with maximum two colours. In this graph, every vertex is a b - vertex. This means that each vertex is a b - vertex for every χ^b - Partition but no vertex is of degree $n - 1 = 3$, as each vertex of C_4 is of degree 2.

Theorem 3.7. If G is a connected with n vertices then $\gamma_{abt}(G) = n$ if and only if $G = K_n$.

Proof: Let G be a connected graph with n vertices. Let V be the vertex set of G . If $G = K_n$ then as $\emptyset(G) = n$, $\gamma_{abt}(G) = n$. Conversely, assume that $\gamma_{abt}(G) = n$. If possible, assume that $G \neq K_n$. So there exist at least two vertices u and v that are not adjacent. Note that $D = V \setminus \{u\}$ is a dominating set.

Claim: $\{u\}$ is not a colour class for every χ^b - Partition

Suppose every χ^b - Partition has $\{u\}$ as colour class. Then $\deg(u) = n - 1$. Clearly, $V \setminus \{v\}$ is a dominating set as well as b - colour transversal. So $\gamma_{abt}(G) \leq n - 1$, which is contradiction to $\gamma_{abt}(G) = n$. Therefore $G = K_n$.

Corollary 3.8. If G is a connected with n vertices then $\gamma_{abt}(G) = n$ if and only if $\emptyset(G) = n$.

Proof: Obvious because for any connected graph G with n vertices, $\emptyset(G) = n$ if and only if $G = K_n$.

Theorem 3.9. If G is a connected graph with n vertices with $\emptyset(G) = n - 1$ then $\gamma_{abt}(G) = n - 1$

Proof: Let G be a connected graph with n vertices.

Suppose $\emptyset(G) = n - 1$. Then as $\emptyset(G) \leq \gamma_{abt}(G)$, $\gamma_{abt}(G) \geq n - 1$. But by earlier theorem, $\gamma_{abt}(G)$ cannot be n . So $\gamma_{abt}(G) = n - 1$.

Remark 3.10. Converse of above theorem 3.9 is not true in general. That is, if G is connected graph with n vertices. Then $\gamma_{abt}(G) = n - 1$ does not imply $\emptyset(G) = n - 1$. See the following graph G with number of vertices $n = 6$ and b - colouring with four colours.

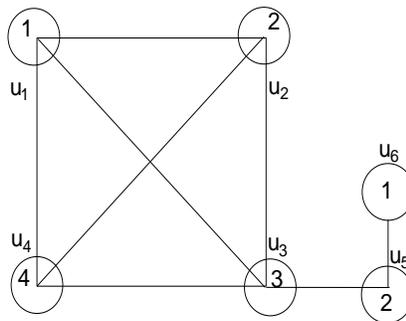


Figure 2: Graph G

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As $m(G) = 2$, $\emptyset(G) = 4 = n - 2$ with b – vertices u_1, u_2, u_3 and u_4 . Here note that u_1, u_2, u_3 and u_4 are the only b – vertices and u_5 & u_6 cannot be b – vertices as their degree is less than 3. Clearly, $\gamma_{abt}(G) = 5 = n - 1$ with γ_{abt} – set $\{u_1, u_2, u_3, u_4, u_5\}$.

Theorem 3.11. If G is a graph with $\gamma(G) = 1$ then $\gamma_{abt}(G) = \emptyset(G)$.

Proof: Let G be a graph with $\gamma(G) = 1$ and $\{v\}$ be a dominating set of G . Note that $\{v\}$ is a colour class for every χ^b – Partition of G . Add one vertex, from each $\emptyset(G) - 1$ colour classes, to the set $\{v\}$. The resultant set will be γ_{abt} – set of cardinality $\emptyset(G)$. Thus $\gamma_{abt}(G) = \emptyset(G)$.

4. Conclusion

It is just the beginning of a talk of this new number ‘dominating b – colour transversal number’. It will be interesting for a researcher to find out this number for path graph, cycle graph, Herschel graph, Grotzsch, Petersen graph etc. B – continuity of b – colouring can also play a vital role in the discussion of some properties of this new number.

Acknowledgements. The author is highly grateful to the reviewers for their careful reading and comments on the article.

Conflicts of Interest. The authors declare no conflicts of interest regarding the publication of this work.

Author’s Contribution. The author solely contributed to the conceptualisation, methodology, analysis, and preparation of the manuscript.

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