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On the Diophantine Equation $31^x + n^y = z^3$

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Abstract. In this research, the solutions of the Diophantine equation $31^x + n^y = z^3$ are investigated where x, y, z are non-negative integers and n is a positive integer such that $n \equiv 5, 6, 25, 26 \pmod{31}$. We show that if $n \equiv 5, 6, 25 \pmod{31}$, then the equation has no solutions. For $n \equiv 26 \pmod{31}$, the solution of the equation is $(x, y, z) = (0, 1, (n+1)^{1/3})$, where $(n+1)^{1/3}$ is a positive integer.

Keywords: Diophantine equation; Integer solution; Congruence; Mihăilescu's Theorem

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1. Introduction

The exponential Diophantine equations of the form $a^x + b^y = z^m$, where a, b, m are fixed positive integers and x, y, z are non-negative integers, have received a lot of attention in recent decades. The Diophantine equations of this type have been studied for $m = 2$. Some of these can be seen in [1], [2], [3] and [4]. Meanwhile, the case of $m = 3$ has also received attention. For example, in 2020, Burshtein [5] studied the Diophantine equation $p^x + q^y = z^3$, where p, q are distinct prime numbers and x, y, z are positive integers such that $x, y \in \{1, 2\}$. After that, in 2022, Mina and Bacani [6] found some sufficient conditions for the non-existence of integer solutions of the Diophantine equation $p^x + (p+4)^y = z^3$, where p and $p+14$ are cousin prime numbers with $p \equiv 1 \pmod 3$. Later, in 2023, Tadee [7] discovered all non-negative integer solutions of the Diophantine equation $8^x + p^y = z^3$, where p is a prime number. In 2024, Tadee [8] presented some conditions for the non-existence of non-negative integer solutions of the Diophantine equation $3^x + n^y = z^3$, where n is a positive integer. In the same year, Khanom et al. [9] solved the Diophantine equation $3^x + a^y = z^3$, where a is a positive integer.

Recently, in 2025, Tadee [10] showed that if a is a positive integer such that $a \equiv 3, 9, 10 \pmod{13}$, then the Diophantine equation $13^x + a^y = z^3$ has no non-negative integer solutions. In 2026, Yiamras and Tadee [11] presented all non-negative integer solutions (x, y, z) of the Diophantine equation $7^x + n^y = z^3$, where n is a positive integer with $n \equiv 2,3,4,5 \pmod 7$. In the same year, Kaewsod and Tadee [12] proved that the Diophantine equation $p^x + n^{3y} = z^3$, when n is a positive integer and $p \notin \{3,7\}$ is a prime number with $n \equiv 1 \pmod p$, has no non-negative integer solutions. Moreover, Tadee [13] also showed that if n is a positive integer such that $n \equiv 2 \pmod{91}$, then the Diophantine equation $13^x + n^y = z^3$ has no non-negative integer solutions.

In this paper, we investigate the Diophantine equation $31^x + n^y = z^3$, where x, y, z are non-negative integers and n is a positive integer with $n \equiv 5, 6, 25, 26 \pmod{31}$.

2. Preliminaries

In this section, we introduce certain a definition and theorems that will be applied throughout this paper.

Definition 2.1. [14] Let $n > 1$ and $\gcd(a,n) = 1$. The order of a modulo n (in older terminology: the exponent to which a belongs modulo n) is the smallest positive integer k such that $a^k \equiv 1 \pmod n$.

Theorem 2.1.[14] Let the integer a have order k modulo n. Then $a^h \equiv 1 \pmod n$ if and only if $k \mid h$; in particular, $k \mid \phi(n)$.

Theorem 2.2. [6] Let p be a prime number and a an integer such that $\gcd(a, p) = 1$. Then, the congruence $x^3 \equiv a \pmod p$ has a solution if and only if $a^{(p-1)/d} \equiv 1 \pmod p$, where $d = \gcd(3, p-1)$. If it has a solution, then there are exactly d solutions modulo p.

Theorem 2.3. [13] Let $p \neq 3$ be a prime number and let n be a positive integer such that $\gcd(n, p) = 1$. If the Diophantine equation $p^x + n^y = z^3$ has a non-negative integer solution (x, y, z) and $y \equiv 0 \pmod 3$, then $1 + 3 \cdot n^{y/3} + 3 \cdot n^{2y/3} = p^x$.

Theorem 2.4. (Catalan's conjecture or Mihăilescu's Theorem) [15] The Diophantine equation $a^x - b^y = 1$, where a, b, x and y are integers such that $\min\{a, b, x, y\} > 1$, has the unique integer solution $(a, b, x, y) = (3, 2, 2, 3)$.

By Theorem 2.4, we have the following corollary.

Corollary 2.5. [9] Let a be a positive integer. The Diophantine equation $1 + a^y = z^3$ has at most one integer solution for fixed a; namely, $(y, z, a) = (1, \sqrt[3]{a+1}, a)$.

Lemma 2.6. Let n be a positive integer and let h be a non-negative integer. Then:
(1) If $n \equiv 5, 25 \pmod{31}$, then $1 + 3 \cdot n^h + 3 \cdot n^{2h} \equiv 7, 29 \pmod{31}$.
(2) If $n \equiv 6, 26 \pmod{31}$, then $1 + 3 \cdot n^h + 3 \cdot n^{2h} \equiv 1, 3, 7, 29, 30 \pmod{31}$.

Proof:
(1) Consider $n \equiv 5 \pmod{31}$. Notice that 3 is the order of 5 modulo 31. By the Division algorithm, there exist two integers q and r such that $h = 3q + r$, when $r \in \{0, 1, 2\}$. Now, we consider the following cases:

Case 1.1. $r = 0$. Then $h = 3q$. It implies that

$$1 + 3\cdot 5^h + 3\cdot 5^{2h} \equiv 1 + 3\cdot 5^{3q} + 3\cdot 5^{6q} \equiv 1 + 3\cdot 1^q + 3\cdot 1^{2q} \equiv 7 \pmod{31}.$$

Case 1.2. $r = 1$. Then $h = 3q + 1$. It implies that

$$1 + 3\cdot 5^h + 3\cdot 5^{2h} \equiv 1 + 3\cdot 5^{3q+1} + 3\cdot 5^{6q+2} \equiv 1 + 3\cdot 5\cdot 1^q + 3\cdot 5^2\cdot 1^{2q} \equiv 29 \pmod{31}.$$

Case 1.3. $r = 2$. Then $h = 3q + 2$. It implies that

$$1 + 3\cdot 5^h + 3\cdot 5^{2h} \equiv 1 + 3\cdot 5^{3q+2} + 3\cdot 5^{6q+4} \equiv 1 + 3\cdot 5^2\cdot 1^q + 3\cdot 5\cdot 1^{2q+1} \equiv 29 \pmod{31}.$$

From the above cases, we have $1 + 3\cdot n^h + 3\cdot n^{2h} \equiv 7, 29 \pmod{31}$. In the same way, we also prove the case $n \equiv 25 \pmod{31}$.

(2) Consider $n \equiv 6 \pmod{31}$. Notice that 6 is the order of 6 modulo 31. By the Division algorithm, there exist two integers q and r such that $h = 6q + r$, when $r \in \{0, 1, 2, 3, 4, 5\}$. Now, we consider the following cases:

Case 2.1. $r = 0$. Then $h = 6q$. It implies that

$$1 + 3\cdot 6^h + 3\cdot 6^{2h} \equiv 1 + 3\cdot 6^{6q} + 3\cdot 6^{12q} \equiv 1 + 3\cdot 1^q + 3\cdot 1^{2q} \equiv 7 \pmod{31}.$$

Case 2.2. $r = 1$. Then $h = 6q + 1$. It implies that

$$1 + 3\cdot 6^h + 3\cdot 6^{2h} \equiv 1 + 3\cdot 6^{6q+1} + 3\cdot 6^{12q+2} \equiv 1 + 3\cdot 6\cdot 1^q + 3\cdot 6^2\cdot 1^{2q} \equiv 3 \pmod{31}.$$

Case 2.3. $r = 2$. Then $h = 6q + 2$. It implies that

$$1 + 3\cdot 6^h + 3\cdot 6^{2h} \equiv 1 + 3\cdot 6^{6q+2} + 3\cdot 6^{12q+4} \equiv 1 + 3\cdot 6^2\cdot 1^q + 3\cdot 6^4\cdot 1^{2q} \equiv 29 \pmod{31}.$$

Case 2.4. $r = 3$. Then $h = 6q + 3$. It implies that

$$1 + 3\cdot 6^h + 3\cdot 6^{2h} \equiv 1 + 3\cdot 6^{6q+3} + 3\cdot 6^{12q+6} \equiv 1 + 3\cdot 6^3\cdot 1^q + 3\cdot 1^{2q+1} \equiv 1 \pmod{31}.$$

Case 2.5. $r = 4$. Then $h = 6q + 4$. It implies that

$$1 + 3\cdot 6^h + 3\cdot 6^{2h} \equiv 1 + 3\cdot 6^{6q+4} + 3\cdot 6^{12q+8} \equiv 1 + 3\cdot 6^4\cdot 1^q + 3\cdot 6^2\cdot 1^{2q+1} \equiv 29 \pmod{31}.$$

Case 2.6. $r = 5$. Then $h = 6q + 5$. It implies that

$$1 + 3\cdot 6^h + 3\cdot 6^{2h} \equiv 1 + 3\cdot 6^{6q+5} + 3\cdot 6^{12q+10} \equiv 1 + 3\cdot 6^5\cdot 1^q + 3\cdot 6^4\cdot 1^{2q+1} \equiv 30 \pmod{31}.$$

Hence, we conclude that $1 + 3\cdot n^h + 3\cdot n^{2h} \equiv 1, 3, 7, 29, 30 \pmod{31}$. In the same way, we also prove the case $n \equiv 26 \pmod{31}$.

3. Main results

In this section, we will consider the Diophantine equation $31^x + n^y = z^3$ for the cases $n \equiv 5 \pmod{31}$, $n \equiv 6 \pmod{31}$, $n \equiv 25 \pmod{31}$ and $n \equiv 26 \pmod{31}$, respectively.

Theorem 3.1. Let n be a positive integer such that $n \equiv 5 \pmod{31}$. Then the Diophantine equation $31^x + n^y = z^3$ has no solutions in non-negative integers x, y and z.

Proof: Suppose, to get a contradiction, that there exist non-negative integers x, y and z such that $31^x + n^y = z^3$. If $x = 0$, then the equation becomes $1 + n^y = z^3$. By Corollary 2.5, we get $z^3 = n + 1$. Since $n \equiv 5 \pmod{31}$, it follows that $z^3 \equiv 6 \pmod{31}$. By Theorem 2.2, we obtain $6^{10} \equiv 1 \pmod{31}$. Since 6 is the order of 6 modulo 31, by Theorem 2.1, we have $6 | 10$. This is impossible. Hence $x > 0$. Then $31^x + n^y \equiv 5^y \pmod{31}$ and so $z^3 \equiv 5^y \pmod{31}$. Since $\gcd(3,30) = 3$, by Theorem 2.2, we get $5^{10y} \equiv 1 \pmod{31}$. Notice that 3 is the order of 5 modulo 31. By Theorem 2.1, it implies that $3 | 10y$. Therefore, we have $3 | y$. So, there exists a non-negative integer h such that $y = 3h$. By Theorem 2.3, we obtain $1 + 3 \cdot n^h + 3 \cdot n^{2h} = 31^x$. It follows that $1 + 3 \cdot n^h + 3 \cdot n^{2h} \equiv 0 \pmod{31}$. This contradicts Lemma 2.6(1).

Example 3.1. The Diophantine equation $31^x + 5^y = z^3$ has no solutions in non-negative integers x, y and z.

Theorem 3.2. Let n be a positive integer such that $n \equiv 6 \pmod{31}$. Then the Diophantine equation $31^x + n^y = z^3$ has no solutions in non-negative integers x, y and z.

Proof: Suppose, to get a contradiction, that there exist non-negative integers x, y and z such that $31^x + n^y = z^3$. If $x = 0$, then the equation becomes $1 + n^y = z^3$. By Corollary 2.5, we get $z^3 = n + 1$. Since $n \equiv 6 \pmod{31}$, it follows that $z^3 \equiv 7 \pmod{31}$. By Theorem 2.2, we obtain $7^{10} \equiv 1 \pmod{31}$. Since 15 is the order of 7 modulo 31, by Theorem 2.1, we have $15 | 10$. This is impossible. Hence $x > 0$. Then $31^x + n^y \equiv 6^y \pmod{31}$ and so $z^3 \equiv 6^y \pmod{31}$. Since $\gcd(3,30) = 3$, by Theorem 2.2, we get $6^{10y} \equiv 1 \pmod{31}$. Notice that 6 is the order of 6 modulo 31. By Theorem 2.1, this implies that $6 | 10y$. Therefore, we have $3 | y$. So, there exists a non-negative integer h such that $y = 3h$. By Theorem 2.3, we obtain $1 + 3 \cdot n^h + 3 \cdot n^{2h} = 31^x$. It follows that $1 + 3 \cdot n^h + 3 \cdot n^{2h} \equiv 0 \pmod{31}$. This contradicts Lemma 2.6(2).

Example 3.2. The Diophantine equation $31^x + 6^y = z^3$ has no solutions in non-negative integers x, y and z.

Theorem 3.3. Let n be a positive integer with $n \equiv 25 \pmod{31}$. Then the Diophantine equation $31^x + n^y = z^3$ has no solutions in non-negative integers x, y and z.

Proof: Suppose, to get a contradiction, that there exist non-negative integers x, y and z such that $31^x + n^y = z^3$. If $x = 0$, then the equation becomes $1 + n^y = z^3$. By Corollary 2.5, we obtain $z^3 = n + 1$. Since $n \equiv 25 \pmod{31}$, it follows that $z^3 \equiv 26 \pmod{31}$. By Theorem 2.2, we have $26^{10} \equiv 1 \pmod{31}$. Since 6 is the order of 26 modulo 31, by Theorem 2.1, it implies that $6 \mid 10$, which is impossible. Hence $x > 0$. Therefore, we get $31^x + n^y \equiv 25^y \pmod{31}$ and so $z^3 \equiv 25^y \pmod{31}$. Since $\gcd(3,30) = 3$, by Theorem 2.2, we obtain $25^{10y} \equiv 1 \pmod{31}$. Notice that 3 is the order of 25 modulo 31. This implies, by Theorem 2.1, that $3 \mid 10y$. Therefore, we have $3 \mid y$. So, there exists a non-negative integer h such that $y = 3h$. By using Theorem 2.3, we obtain $1 + 3 \cdot n^h + 3 \cdot n^{2h} = 31^x$. Consequently, $1 + 3 \cdot n^h + 3 \cdot n^{2h} \equiv 0 \pmod{31}$. This contradicts Lemma 2.6(1).

Example 3.3. The Diophantine equation $31^x + 25^y = z^3$ has no solutions in non-negative integers x, y and z.

Theorem 3.4. Let n be a positive integer with $n \equiv 26 \pmod{31}$. Then the non-negative integer solution of the Diophantine equation $31^x + n^y = z^3$ is $(x, y, z) = (0, 1, \sqrt[3]{n+1})$, where $\sqrt[3]{n+1}$ is a positive integer.

Proof: Assume that $x > 0$. Take the Diophantine equation $31^x + n^y = z^3$ modulo 31 to get $z^3 \equiv 26^y \pmod{31}$. Since $\gcd(3,30) = 3$, by Theorem 2.2, we have $26^{10y} \equiv 1 \pmod{31}$. Notice that 6 is the order of 26 modulo 31. This implies, by Theorem 2.1, that $6 \mid 10y$. Therefore, we have $3 \mid y$. So, there exists a non-negative integer h such that $y = 3h$. By Theorem 2.3, we get $1 + 3 \cdot n^h + 3 \cdot n^{2h} = 31^x$. It follows that $1 + 3 \cdot n^h + 3 \cdot n^{2h} \equiv 0 \pmod{31}$. This contradicts Lemma 2.6(2). Thus, we get $x = 0$. The equation becomes $1 + n^y = z^3$. By Corollary 2.5, it implies that $(x, y, z) = (0, 1, \sqrt[3]{n+1})$, where $\sqrt[3]{n+1}$ is a positive integer.

Example 3.4. The Diophantine equation $31^x + 26^y = z^3$ has the unique non-negative integer solution $(x, y, z) = (0, 1, 3)$.

Example 3.5. The Diophantine equation $31^x + 57^y = z^3$ has no solutions in non-negative integers x, y and z.

3. Conclusion

In this article, we consider the Diophantine equation $31^x + n^y = z^3$, where x, y, z are non-negative integers and n is a positive integer such that $n \equiv 5, 6, 25, 26 \pmod{31}$. Using a combination of modular arithmetic and Mihăilescu's Theorem, we demonstrate that if $n \equiv 5, 6, 25 \pmod{31}$, then the equation is unsolvable. For the case $n \equiv 26 \pmod{31}$, the non-negative integer solution (x, y, z) of the equation is $(0, 1, \sqrt[3]{n+1})$, where $\sqrt[3]{n+1}$ is a positive integer.

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