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On Some Generalized Identities of Dickson *k*-Fibonacci and *k*-Lucas Polynomials

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Abstract. In this paper, we establish several generalized identities for Dickson *k*-Fibonacci and Dickson *k*-Lucas polynomials, which were introduced by Engin Ozkan and Hakan Akkus in 2024. The identities are derived using Binet's formula and the generating function. We also obtain the exponential Generating function and alternating Convolution identity for these polynomials.

Keywords: *k*-Fibonacci polynomials; *k*-Lucas polynomials; Dickson polynomials; Binet formula; Generating function.

AMS Mathematics Subject Classification (2020): 11B39; 11B37

1. Introduction

Polynomials play a fundamental role in mathematics because they provide simple yet powerful tools for modeling, approximation, and computation [8,10,12]. They are widely applied in algebra, numerical analysis, differential equations, computer graphics, coding theory, cryptography, signal processing, control systems, optimization, machine learning, physics, engineering, economics, and various branches of scientific and technological research. The Fibonacci and Lucas sequences have long played a significant role in mathematics. They have found numerous applications in fields such as number theory, combinatorics, physics, biology, and computer science have all used it. It seems to be an obsession for researchers. Over time, mathematicians have created a number of variations, such as the *k*-Fibonacci and *k*-Lucas numbers, where a parameter modifies the recurrence. Meanwhile, by replacing normal coefficients with polynomial expressions, Fibonacci and Lucas polynomials can be used to generate a number of fascinating algebraic patterns.

The Fibonacci number sequence [19], is defined by the recurrence relation

$$F_n = F_{n-1} + F_{n-2}\,,\ n \geq 2.\ with\ F_0 = 0, F_1 = 1 \tag{1}$$

The Lucas number sequence [19], is defined by the recurrence relation

$$L_n = L_{n-1} + L_{n-2}\,,\; n \geq 2.\; with\; L_0 = 2, L_1 = 1 \quad (2)$$

The k-Fibonacci numbers defined by Falcon and Plaza [14,15,16 and 17], for any positive real number k, the k-Fibonacci sequence is defined recurrently by

$$F_{k,n} = kF_{k,n-1} + F_{k,n-2}\,,\; n \geq 2.\; with\; F_{k,0} = 0, F_{k,1} = 1 \quad (3)$$

The k-Lucas numbers defined by Falcon [13],

$$L_{k,n} = kL_{k,n-1} + L_{k,n-2}\,,\; n \geq 2.\; with\; L_{k,0} = 2, L_{k,1} = k \quad (4)$$

Most of the authors introduced Fibonacci pattern-based polynomials in many ways which are known as Generalized Fibonacci polynomials [1, 2, 3, 9], Generalized Fibonacci-type polynomials [5, 6, 20, 21]. Now, Dickson polynomials also hold a distinct significance of their own. Leonard Eugene Dickson introduced them in 1897, and ever since, they have assumed immense importance in fields such as cryptography, coding theory, and the study of permutation polynomials over finite fields. Their most prominent feature is that they have several strong features and satisfy a second-order linear recurrence relation; as a result, they are particularly helpful in solving both real-world and theoretical mathematical issues.

For $n \geq 2$ and α with identity in a commutative ring $\mathcal{R}$, Dickson polynomials first kind $D_n(x,\alpha)$ are defined recurrently by

$$D_n(x,\alpha) = xD_{n-1}(x,\alpha) - \alpha\, D_{n-2}(x,\alpha)\;\; with\; D_0 = 2, D_1 = x \quad (5)$$

Dickson polynomials of the second kind $E_n(x,\alpha)$ are defined recurrently by

$$E_n(x,\alpha) = xE_{n-1}(x,\alpha) - \alpha\, E_{n-2}(x,\alpha)\; with\; E_0 = 1, E_1 = x \quad (6)$$

Dickson k-Fibonacci and Dickson k-Lucas polynomials have significant applications in cryptography and coding theory, particularly through their connection to permutation polynomials over finite fields. These polynomials are used to construct Dembowski-Ostrom (DO) polynomials and planar functions, which are essential for designing S-boxes in block cyphers and public-key cryptosystems like HFE [11, 18].

2. Preliminaries

Before presenting our main theorems, we need to introduce some known results and notations.

The Dickson k-Fibonacci polynomials defined by Ozkan and Akkus [4], for $k \in \mathbb{R}^+$ and $n \in \mathbb{N}$, the Dickson k-Fibonacci polynomials is defined recurrently by

$$D\mathcal{F}_{k,n+2}(x) = kxD\mathcal{F}_{k,n+1}(x) + D\mathcal{F}_{k,n}(x) \quad (7)$$

with $D\mathcal{F}_{k,0}(x) = 0$ and $D\mathcal{F}_{k,1}(x) = x$.

The Dickson k-Lucas polynomials defined by Akkus and Ozkan [7], for $k \in \mathbb{R}^+$ and $n \in \mathbb{N}$, the Dickson k-Lucas polynomials is defined recurrently by

$$D\mathcal{L}_{k,n+2}(x) = kxD\mathcal{L}_{k,n+1}(x) + D\mathcal{L}_{k,n}(x) \quad (8)$$

with $D\mathcal{L}_{k,0}(x) = 2$ and $D\mathcal{L}_{k,1}(x) = kx$.

First few terms of the polynomials are shown in the following Table 1:

n	$D\mathcal{F}_{k,n}(x)$	$D\mathcal{L}_{k,n}(x)$
0	0	2
1	1	kx
2	kx	k^2x^2+2
3	k^2x^2	k^3x^3+3kx
4	k^3x^3+2kx	$k^4x^4+4k^2x^2+2$
⋮	⋮	⋮

Table 1: Dickson *k*-Fibonacci and Dickson *k*-Lucas polynomials

The characteristic roots of recurrence relation (7) and (8) are $c = \frac{kx+\sqrt{k^2x^2+4}}{2}$ and $d = \frac{kx-\sqrt{k^2x^2+4}}{2}$. Note that $c+d = kx,\ cd = -1, c-d = \sqrt{k^2x^2+4}$
Binet's formula for the Dickson *k*-Fibonacci and Dickson *k*-Lucas polynomials are given by

$$D\mathcal{F}_{k,n}(x) = \frac{c^n-d^n}{c-d} \tag{9}$$

$$D\mathcal{L}_{k,n}(x) = c^n + d^n \tag{10}$$

Generating function for the Dickson *k*-Fibonacci and Dickson *k*-Lucas polynomials are given by

$$\sum_{n=0}^{\infty} D\mathcal{F}_{k,n}(x)\, t^n = \frac{t}{1-tkx-t^2} \tag{11}$$

$$\sum_{n=0}^{\infty} D\mathcal{L}_{k,n}(x)\, t^n = \frac{2-tkx}{1-tkx-t^2} \tag{12}$$

3. Results

In this section, we present some generalized identities for Dickson *k*-Fibonacci and Dickson *k*-Lucas polynomials, from which we obtain Catlan's identity, Cassini's identity and d'Ocagne's identity.

Theorem 3.1. (Generalized identity). For $n > m \geq b \geq 1$,

$$D\mathcal{F}_{k,m}(x)D\mathcal{F}_{k,n}(x) - D\mathcal{F}_{k,m-b}(x)D\mathcal{F}_{k,n+b}(x) = (-1)^{m-b}D\mathcal{F}_{k,b}(x)D\mathcal{F}_{k,n-m+b}(x) \tag{13}$$

Proof: By Binet's formula

$$D\mathcal{F}_{k,m}(x)D\mathcal{F}_{k,n}(x) - D\mathcal{F}_{k,m-b}(x)D\mathcal{F}_{k,n+b}(x)$$

$$= \left(\frac{c^m-d^m}{c-d}\right)\left(\frac{c^n-d^n}{c-d}\right) - \left(\frac{c^{m-b}-d^{m-b}}{c-d}\right)\left(\frac{c^{n+b}-d^{n+b}}{c-d}\right)$$

$$= \frac{c^m d^n(c^{-b}d^b-1) + d^m c^n(c^b d^{-b}-1)}{(c-d)^2}$$

$$= \frac{(-1)^m}{(c-d)^2}\left(\frac{d^{n-m+b}(d^b-c^b) + c^{n-m+b}(c^b-d^b)}{c^b d^b}\right)$$

$$= \frac{(-1)^{m-b}}{(c-d)^2}(c^b-d^b)(c^{n-m+b}-d^{n-m+b})$$

$$= (-1)^{m-b}D\mathcal{F}_{k,b}(x)D\mathcal{F}_{k,n-m+b}(x)$$

This completes the proof. □

Corollary 3.1.1. (Catlan's identity). If we consider $n = m$ in the generalized identity (13), we obtain

$$D\mathcal{F}_{k,n}^2(x) - D\mathcal{F}_{k,n-b}(x)D\mathcal{F}_{k,n+b}(x) = (-1)^{n-b}D\mathcal{F}_{k,b}^2(x) \quad (14)$$

Corollary 3.1.2. (Cassini's identity). If we consider $n = m$ and $b = 1$ in the generalized identity (13), we obtain

$$D\mathcal{F}_{k,n}^2(x) - D\mathcal{F}_{k,n-1}(x)D\mathcal{F}_{k,n+1}(x) = (-1)^{n-1} \quad (15)$$

Corollary 3.1.3. (d'Ocagne's identity). If we consider $n = m$, $m = n + 1$ and $b = 1$ in the generalized identity (13), we obtain

$$D\mathcal{F}_{k,m}(x)D\mathcal{F}_{k,n+1}(x) - D\mathcal{F}_{k,m+1}(x)D\mathcal{F}_{k,n}(x) = (-1)^{n}D\mathcal{F}_{k,m-n}(x) \quad (16)$$

Theorem 3.2. (Generalized identity). For $n > m \geq b \geq 1$,

$$D\mathcal{L}_{k,m}(x)D\mathcal{L}_{k,n}(x) - D\mathcal{L}_{k,m-b}(x)D\mathcal{L}_{k,n+b}(x)$$
$$= (-1)^{1-b}(k^2x^2 + 4)D\mathcal{F}_{k,b}(x)D\mathcal{F}_{k,n-m+b}(x) \quad (17)$$

Proof: By Binet's formula

$$D\mathcal{L}_{k,m}(x)D\mathcal{L}_{k,n}(x) - D\mathcal{L}_{k,m-b}(x)D\mathcal{L}_{k,n+b}(x)$$
$$= (c^m + d^m)\,(c^n + d^n) - \left(c^{m-b} + d^{m-b}\right)\left(c^{n+b} + d^{n+b}\right)$$
$$= c^m d^n\left(1 - \frac{d^b}{c^b}\right) + \ d^m c^n\left(1 - \frac{c^b}{d^b}\right)$$
$$= d^{n-m}\left(\frac{c^b - d^b}{c^b}\right) + c^{n-m}\left(\frac{d^b - c^b}{d^b}\right)$$
$$= -(cd)^{-b}\left\{c^{n-m+b}\left(c^b - d^b\right) - d^{n-m+b}\left(c^b - d^b\right)\right\}$$
$$= (-1)^{1-b}(k^2x^2 + 4)\left(\frac{c^b - d^b}{c - d}\right)\left(\frac{c^{m-n-b} - d^{m-n-b}}{c - d}\right)$$
$$= (-1)^{1-b}(k^2x^2 + 4)D\mathcal{F}_{k,b}(x)D\mathcal{F}_{k,n-m+b}(x)$$

This completes the proof. □

Corollary 3.4. (Catlan's identity). If we consider $n = m$ in the generalized identity (17), we obtain

$$D\mathcal{L}_{k,n}^2(x) - D\mathcal{L}_{k,n-b}(x)D\mathcal{L}_{k,n+b}(x) = (-1)^{1-b}(k^2x^2 + 4)D\mathcal{F}_{k,b}^2(x) \quad (18)$$

Corollary 3.5. (Cassini's identity). If we consider $n = m$ and $b = 1$ in the generalized identity (17), we obtain

$$D\mathcal{L}_{k,n}^2(x) - D\mathcal{L}_{k,n-1}(x)D\mathcal{L}_{k,n+1}(x) = (k^2x^2 + 4) \quad (19)$$

Corollary 3.6. (d'Ocagne's identity). If we consider $n = m$, $m = n + 1$ and $b = 1$ in the generalized identity (17), we obtain

$$D\mathcal{L}_{k,m}(x)D\mathcal{L}_{k,n+1}(x) - D\mathcal{L}_{k,m+1}(x)D\mathcal{L}_{k,n}(x) = (k^2x^2 + 4)D\mathcal{F}_{k,m-n}(x) \quad (20)$$

Theorem 3.3.

$$D\mathcal{F}_{k,m+n}(x)D\mathcal{F}_{k,m+b}(x) - D\mathcal{F}_{k,m}(x)D\mathcal{F}_{k,m+n+b}(x) = (-1)^{m}D\mathcal{F}_{k,b}(x)D\mathcal{F}_{k,n-m+b}(x)$$

Proof: By Binet's formula

$$D\mathcal{F}_{k,m+n}(x)D\mathcal{F}_{k,m+b}(x) - D\mathcal{F}_{k,m}(x)D\mathcal{F}_{k,m+n+b}(x)$$
$$= \left(\frac{c^{m+n} - d^{m+n}}{c-d}\right)\left(\frac{c^{m+b} - d^{m+b}}{c-d}\right) - \left(\frac{c^{m} - d^{m}}{c-d}\right)\left(\frac{c^{m+n+b} - d^{m+n+b}}{c-d}\right)$$
$$= \frac{(c^b - d^b)}{(c-d)^2}\left\{c^{m+b}\left(\frac{-1}{c}\right)^m - d^{m+b}\left(\frac{-1}{d}\right)^m\right\}$$
$$= (-1)^m\left(\frac{c^n - d^n}{c-d}\right)\left(\frac{c^b - d^b}{c-d}\right)$$
$$= (-1)^m D\mathcal{F}_{k,b}(x)D\mathcal{F}_{k,n-m+b}(x)$$

This completes the proof. □

Theorem 3.4.

$$D\mathcal{F}_{k,m}(x)D\mathcal{F}_{k,n+1}(x) + D\mathcal{F}_{k,m-1}(x)D\mathcal{F}_{k,n}(x)$$
$$= D\mathcal{L}_{k,m+n+1}(x) - D\mathcal{L}_{k,m+n-1}(x) - 2(-1)^m D\mathcal{L}_{k,n-m+1}(x)$$

Proof: By Binet's formula

$$D\mathcal{F}_{k,m}(x)D\mathcal{F}_{k,n+1}(x) + D\mathcal{F}_{k,m-1}(x)D\mathcal{F}_{k,n}(x)$$

$$= \left(\frac{c^{m} - d^{m}}{c-d}\right)\left(\frac{c^{n+1} - d^{n+1}}{c-d}\right) - \left(\frac{c^{m-1} - d^{m-1}}{c-d}\right)\left(\frac{c^{n} - d^{n}}{c-d}\right)$$

$$= c^{m+n}(c - c^{-1}) + d^{m+n}(d - d^{-1}) + c^m d^n(c^{-1} - d) + d^m c^n(d^{-1} - c)$$

$$= c^{m+n-1}(c^2 - 1) + d^{m+n-1}(d^2 - 1) + c^{m-1}d^n(1 - cd) + d^{m-1}c^n(1 - cd)$$

$$= (c^{m+n+1} + d^{m+n+1}) - (c^{m+n-1} + d^{m+n-1}) - 2(-1)^m(c^{n-m+1} + d^{n-m+1})$$

$$= D\mathcal{L}_{k,m+n+1}(x) - D\mathcal{L}_{k,m+n-1}(x) - 2(-1)^m D\mathcal{L}_{k,n-m+1}(x)$$

This completes the proof. □

Theorem 3.5.

$$\sum_{i=0}^{n} D\mathcal{F}_{k,pi+q}(x)$$
$$= \begin{cases} \dfrac{(-1)^p D\mathcal{F}_{k,pn+q}(x) - D\mathcal{F}_{k,pn+q+q}(x) + D\mathcal{F}_{k,q-1}(x) + (-1)^q D\mathcal{F}_{k,p-q}(x)}{(-1)^p - D\mathcal{L}_{k,p}(x) + 1}, if\ q < p \\ \dfrac{(-1)^p D\mathcal{F}_{k,pn+q}(x) - D\mathcal{F}_{k,pn+q+q}(x) + D\mathcal{F}_{k,q-1}(x) - (-1)^p D\mathcal{F}_{k,q-p}(x)}{(-1)^p - D\mathcal{L}_{k,p}(x) + 1}, \text{otherwise} \end{cases}$$

Proof: By Binet's formula

$$\sum_{i=0}^{n} D\mathcal{F}_{k,pi+q}(x) = \sum_{i=0}^{n}\left(\frac{c^{pi+q} - d^{pi+q}}{c-d}\right)$$

$$= \frac{1}{c-d}\sum_{i=0}^{n}\left(c^{pi+q} - d^{pi+q}\right)$$
$$= \frac{1}{c-d}\left(\frac{c^{pn+p+q} - c^{q}}{c^{p}-1} - \frac{d^{pn+p+q} - d^{q}}{d^{p}-1}\right)$$
$$= \frac{1}{(c-d)(-1)^{p} - D\mathcal{L}_{k,p}(x)+1}\left(\frac{(cd)^{p}(c^{pn+q}-d^{pn+q})-(c^{pn+q+q}-d^{pn+q+q})+(c^{q}-d^{q})+(cd)^{q}(c^{p-q}-d^{p-q})}{(-1)^{p}-D\mathcal{L}_{k,p}(x)+1}\right)$$

$$= \begin{cases} \dfrac{(-1)^{p}D\mathcal{F}_{k,pn+q}(x) - D\mathcal{F}_{k,pn+q+q}(x) + D\mathcal{F}_{k,q}(x) + (-1)^{q}D\mathcal{F}_{k,p-q}(x)}{(-1)^{p} - D\mathcal{L}_{k,p}(x)+1}, if\ q < p \\ \dfrac{(-1)^{p}D\mathcal{F}_{k,pn+q}(x) - D\mathcal{F}_{k,pn+q+q}(x) + D\mathcal{F}_{k,q}(x) - (-1)^{p}D\mathcal{F}_{k,q-p}(x)}{(-1)^{p} - D\mathcal{L}_{k,p}(x)+1}, otherwise \end{cases}$$

This completes the proof. □

Theorem 3.6.

$$\sum_{i=0}^{n} D\mathcal{L}_{k,pi+q}(x)$$
$$= \begin{cases} \dfrac{(-1)^{p}D\mathcal{L}_{k,pn+q}(x) - D\mathcal{L}_{k,pn+q+q}(x) + D\mathcal{L}_{k,q}(x) + (-1)^{q}D\mathcal{L}_{k,p-q}(x)}{(-1)^{p} - D\mathcal{L}_{k,p}(x)+1}, if\ q < p \\ \dfrac{(-1)^{p}D\mathcal{L}_{k,pn+q}(x) - D\mathcal{L}_{k,pn+q+q}(x) + D\mathcal{L}_{k,q-1}(x) - (-1)^{p}D\mathcal{L}_{k,q-p}(x)}{(-1)^{p} - D\mathcal{L}_{k,p}(x)+1}, \text{otherwise} \end{cases}$$

Proof: By Binet's formula

$$\sum_{i=0}^{n} D\mathcal{L}_{k,pi+q}(x) = \sum_{i=0}^{n}\left(c^{pi+q} + d^{pi+q}\right)$$
$$= \left(\frac{c^{pn+p+q} - c^{q}}{c^{p}-1} + \frac{d^{pn+p+q} - d^{q}}{d^{p}-1}\right)$$

$$= \left(\frac{(cd)^{p}(c^{pn+q}+d^{pn+q}) - (c^{pn+q+q}+d^{pn+q+q}) + (c^{q}+d^{q}) + (cd)^{q}(c^{p-q}+d^{p-q})}{(-1)^{p} - D\mathcal{L}_{k,p}(x)+1}\right)$$

$$= \begin{cases} \dfrac{(-1)^{p}D\mathcal{L}_{k,pn+q}(x) - D\mathcal{L}_{k,pn+q+q}(x) + D\mathcal{L}_{k,q}(x) + (-1)^{q}D\mathcal{L}_{k,p-q}(x)}{(-1)^{p} - D\mathcal{L}_{k,p}(x)+1}, if\ q < p \\ \dfrac{(-1)^{p}D\mathcal{L}_{k,pn+q}(x) - D\mathcal{L}_{k,pn+q+q}(x) + D\mathcal{L}_{k,q-1}(x) - (-1)^{p}D\mathcal{L}_{k,q-p}(x)}{(-1)^{p} - D\mathcal{L}_{k,p}(x)+1}, \text{otherwise} \end{cases}$$

This completes the proof. □

4. Generating functions for Dickson *k*-Fibonacci and *k*-Lucas polynomials

In this section, we establish some identities for Dickson *k*-Fibonacci and Dickson *k*-Lucas polynomials. Generating functions are very helpful in finding the relations for the sequences of integers. Many authors found miscellaneous identities for Fibonacci and

Lucas polynomials by manipulation with their generating functions. Our approach is rather different in this section.

Theorem 4.1. For $b \in \mathbb{Z}$, we have

$$\sum_{m=0}^{b} D\mathcal{F}_{k,m}(x)\, t^{-m} = \frac{1}{(t^2-tkx-1)t^b}\left(t^{b+1} - tD\mathcal{F}_{k,b+1}(x) - D\mathcal{F}_{k,b}(x)\right) \qquad (21)$$

Proof: By Binet's formula

$$\sum_{m=0}^{b} D\mathcal{F}_{k,m}(x)\, t^{-n} = \sum_{m=0}^{b}\left(\frac{c^m - d^m}{c-d}\right) t^{-n}$$

$$= \frac{1}{c-d}\sum_{m=0}^{b}\left\{\left(\frac{c}{t}\right)^m - \left(\frac{d}{t}\right)^m\right\}$$

$$= \frac{1}{c-d}\left\{\frac{1-\left(\frac{c}{t}\right)^{b+1}}{1-\frac{c}{t}} - \frac{1-\left(\frac{d}{t}\right)^{b+1}}{1-\frac{d}{t}}\right\}$$

$$= \frac{1}{(c-d)t^b}\left(\frac{t^{b+1}-c^{b+1}}{t-c} - \frac{t^{b+1}-d^{b+1}}{t-d}\right)$$

$$= \frac{1}{(c-d)t^b}\left(\frac{t^{b+1}(c-d) - t(c^{b+1}-d^{b+1}) - (c^b - d^b)}{(t-c)(t-d)}\right)$$

$$= \frac{1}{(t^2-tkx-1)t^b}\left(t^{b+1} - tD\mathcal{F}_{k,b+1}(x) - D\mathcal{F}_{k,b}(x)\right)$$

This completes the proof. □

Corollary 4.1.1. For $b \to \infty$, we get $\sum_{m=0}^{\infty} D\mathcal{F}_{k,m}(x)\, t^{-m} = \frac{t}{(t^2-tkx-1)}$ (22)

Theorem 4.2. For $b \in \mathbb{Z}$, we have

$$\sum_{m=0}^{b} D\mathcal{L}_{k,m}(x)\, t^{-m} = \frac{2t^2-kxt}{t^2-kxt-1} - \frac{1}{t^b(t^2-kxt-1)}\{tD\mathcal{L}_{k,b+1}(x) + D\mathcal{L}_{k,b}(x)\} \quad (23)$$

Proof: By Binet's formula

$$\sum_{m=0}^{b} D\mathcal{L}_{k,m}(x)\, t^{-m} = \sum_{m=0}^{b}(c^n + d^n)\, t^{-m}$$

$$= \sum_{m=0}^{b}\left\{\left(\frac{c}{t}\right)^m + \left(\frac{d}{t}\right)^m\right\}$$

$$= \frac{1-\left(\frac{c}{t}\right)^{b+1}}{1-\frac{c}{t}} + \frac{1-\left(\frac{d}{t}\right)^{b+1}}{1-\frac{d}{t}}$$

$$= \frac{1}{t^b}\left(\frac{t^{b+1}-c^{b+1}}{t-c}+\frac{t^{b+1}-d^{b+1}}{t-d}\right)$$

$$= \frac{2t^{b+2}-t(c^{b+1}+d^{b+1})-\ t^{b+1}(c+d)+cd(c^b+d^b)}{t^b(t-c)(t-d)}$$

$$= \frac{2t^2-kxt}{t^2-kxt-1}-\frac{1}{t^b(t^2-kxt-1)}\{tD\mathcal{L}_{k,b+1}(x)+D\mathcal{L}_{k,b}(x)\}$$

This completes the proof.

Corollary 4.2. For $b \to \infty$, we get $\sum_{m=0}^{\infty} D\mathcal{L}_{k,m}(x)\, t^{-m} = \frac{2t^2-kxt}{t^2-kxt-1}$ (24)

5. Exponential generating functions of Dickson *k*-Fibonacci and Dickson *k*-Lucas polynomials

In this section, we establish exponential generating functions for Dickson *k*-Fibonacci and Dickson *k*-Lucas polynomials.

Theorem 5.1. $\sum_{n=0}^{\infty} D\mathcal{F}_{k,n}(x)\ \frac{t^n}{n!} = \frac{2e^{\frac{kxt}{2}}\sinh\left(\frac{t}{2}\sqrt{k^2x^2+4}\right)}{\sqrt{k^2x^2+4}}$ (25)

Proof: By Binet's formula

$$D\mathcal{F}_{k,n}(x) = \left(\frac{c^n-d^n}{c-d}\right) = \left(\frac{c^n-d^n}{\Delta}\right)$$

Let $E(t) = \sum_{n=0}^{\infty} D\mathcal{F}_{k,n}(x)\ \frac{t^n}{n!} = \sum_{n=0}^{\infty}\left(\frac{c^n-d^n}{\Delta}\right)\frac{t^n}{n!}$

$$= \frac{1}{\Delta}\left(\sum_{n=0}^{\infty}\frac{(ct)^n}{n!}-\sum_{n=0}^{\infty}\frac{(dt)^n}{n!}\right)$$

$$= \frac{1}{\Delta}\left(e^{ct}-e^{dt}\right)$$

$$= \frac{1}{\Delta}e^{\frac{kxt}{2}}\left(e^{\frac{\Delta t}{2}}-e^{\frac{-\Delta t}{2}}\right)$$

$$= \frac{1}{\Delta}2e^{\frac{kxt}{2}}\sinh\left(\frac{\Delta t}{2}\right)$$

Thus:

$$E(t) = \sum_{n=0}^{\infty} D\mathcal{F}_{k,n}(x)\ \frac{t^n}{n!} = \frac{2e^{\frac{kxt}{2}}\sinh\left(\frac{t}{2}\sqrt{k^2x^2+4}\right)}{\sqrt{k^2x^2+4}}$$

This completes the proof. □

Theorem 5.2. $\sum_{n=0}^{\infty} D\mathcal{L}_{k,n}(x)\ \frac{t^n}{n!} = 2e^{\frac{kxt}{2}}cosh\left(\frac{t}{2}\sqrt{k^2x^2+4}\right)$ (26)

Proof: The proof is clear by Binet's formula. □

6. Alternating convolution identity for Dickson *k*-Fibonacci and Dickson *k*-Lucas polynomials

In this section, we establish an alternating convolution identity for Dickson *k*-Fibonacci and Dickson *k*-Lucas polynomials.

Theorem 6.1. (Alternating Convolution Identity)
For every integer $n \geq 1$and $x \neq 0$,
$\sum_{j=0}^{n}(-1)^j\, D\mathcal{F}_{k,n}(x)\, D\mathcal{L}_{k,n-j}(x) = D\mathcal{F}_{k,n}(x)\cdot\frac{1+(-1)^n}{2} - \frac{1+(-1)^{n+1}}{kx}\cdot D\mathcal{F}_{k,n+1}(x)$
Proof: Let $\mathcal{S}_{k,n} = \sum_{j=0}^{n}(-1)^j\, D\mathcal{F}_{k,n}(x)\, D\mathcal{L}_{k,n-j}(x)$
By Binet formula of Dickson *k*-Fibonacci and Dickson *k*-Lucas polynomials

$$\mathcal{S}_{k,n} = \sum_{j=0}^{n}(-1)^j\left(\frac{c^n - d^n}{c-d}\right)\left(c^{n-j} + d^{n-j}\right)$$
$$\mathcal{S}_{k,n} = \frac{1}{c-d}\left[(c^n - d^n)\sum_{j=0}^{n}(-1)^j + \sum_{j=0}^{n}(-1)^j c^j d^{n-j} - \sum_{j=0}^{n}(-1)^j d^j c^{n-j}\right]$$
$$= \frac{1}{c-d}\left[(c^n - d^n)\cdot\frac{1+(-1)^n}{2} + \frac{d^{n+1}}{kx}(1 - c^{2n+2}) - \frac{c^{n+1}}{kx}(1 - d^{2n+2})\right]$$
$$= \frac{1}{(c-d)}\left[(c-d)(c^n - d^n)\cdot\frac{1+(-1)^n}{2} - \frac{(c-d)(1+(-1)^{n+1})}{kx}(c^{2n+1} - d^{2n+1})\right]$$
$$\mathcal{S}_{k,n} = D\mathcal{F}_{k,n}(x)\cdot\frac{1+(-1)^n}{2} - \frac{1+(-1)^{n+1}}{kx}\cdot D\mathcal{F}_{k,n+1}(x)$$

This completes the proof. □

Corollary 6.1.1. (Even index)
If n is even: $\sum_{j=0}^{n}(-1)^j\, D\mathcal{F}_{k,n}(x)\, D\mathcal{L}_{k,n-j}(x) = D\mathcal{F}_{k,n}(x)$ (28)

Corollary 6.1.2. (Odd index)
If n is odd: $\sum_{j=0}^{n}(-1)^j\, D\mathcal{F}_{k,n}(x)\, D\mathcal{L}_{k,n-j}(x) = -\frac{2}{kx}D\mathcal{F}_{k,n+1}(x)$ (29)

7. Conclusion

In this study, we have stated and derived many fundamental identities of Dickson k-Fibonacci and Dickson k-Lucas polynomials. These identities can be used to develop new polynomial identities. Also, we present exponential generating functions and alternating convolution identities for them. These identities can be used to construct Lucas-based cryptosystems, which are alternatives to RSA and Diffie-Hellman, and the factorisation properties of these polynomials make them suitable for elliptic curve cryptography (ECC) when evaluated at a specific point; also, the roots of these polynomials over finite fields give primitive elements that generate code words.

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