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The Upper Bounds of the Narumi-Katayama Index and Some Hamiltonian Properties of a Graph

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Abstract. The Narumi-Katayama index of a graph is defined as the product of the degrees of all vertices in the graph. Using Schweitzer's inequality, we in this paper obtain upper bounds of the Narumi-Katayama index of a connected graph. We further present sufficient conditions based on the Narumi-Katayama index for Hamiltonian and traceable graphs.

Keywords: Narumi-Katayama index, Hamiltonian graph, traceable graph

AMS Mathematics Subject Classification (2010): 05C45

1. Introduction

We consider only finite undirected graphs without loops or multiple edges. Notation and terminology not defined here follow those in [1]. For a graph G = (V(G), E(G)), we use n and e to denote its order |V(G)| and size |E(G)|, respectively. The minimum degree and maximum degree of G are denoted by $\delta(G)$ and $\Delta(G)$, respectively. The neighborhood of a vertex u in G is denoted by $N_G(u)$. If a vertex $u \in V - S$, where S is a subset of the vertex set V, we define $d_{G[S]}(u)$ as $|N_G(u) \cap S|$, where G[S] is a subgraph of G which is induced by S. A subset T of the vertex set V(G) of G is independent if no two vertices in T are adjacent in G. A maximum independent set in a graph G is an independent set of largest possible size. The independence number, denoted $\alpha(G)$, of a graph G is the cardinality of a maximum independent set in G. For disjoint vertex subsets Y and Z of V(G), We define E(Y, Z) as $\{e : e = yz \in E, y \in Y, z \in Z\}$. For two disjoint graphs G_1 and G_2, we use $G_1 \vee G_2$ to denote the join of G_1 and G_2. We use K_s to denote a complete graph of order s. We also use $K_{s,\, t}$ to denote a complete bipartite graph with two partition sets Y and Z such that |Y| = s and |Z| = t. The complement of a graph H is denoted by H^c. A cycle C in a graph G is called a Hamiltonian cycle of G if C contains all the vertices of G. A graph G is called Hamiltonian if G has a Hamiltonian cycle. A path P in a graph G is called a Hamiltonian path of G if P contains all the vertices of G. A graph G is called traceable if G has a Hamiltonian path.

Narumi and Katayama in [12] introduced the concept of "simple topological index" for a graph and it was defined as the product of the degrees of all vertices in G. In the later works, researchers have used the ''Narumi-Katayama index'' for the "simple topological index" of a graph. See, for instance, [5]. We in this paper will also use the term of Narumi-Katayama yama index and denote the Narumi-Katayama index of a graph G by NK(G). The readers are referred to [4], [5], [6], [7], [11], [12], [14], [15], and [16] to find results on the Narumi-Katayama index and its applications. Very recently, Li [8] obtained upper bounds for the Narumi-Katayama index and presented sufficient conditions based on the Narumi-Katayama index for Hamiltonian and traceable graphs. In this paper, we use Schweitzer's inequality to obtain new upper bounds of the Narumi-Katayama index of a connected graph. We further present sufficient conditions based on the Narumi-Katayama index for Hamiltonian and traceable graphs. Below are the results of this paper.

Theorem 1.1. Let G be a connected graph of order $n \geq 2$ vertices and e edges. Then [1] $NK(G) \leq (n - \alpha)^{\alpha} \left(\frac{(\delta + \Delta)^2}{4\delta}\right)^{n - \alpha}$ with equality if and only if G is a regular graph such that $d(x) = n - \alpha$ for each $x \in V$, $d_{G[I]}(z) = \alpha$ and $d_{G[V - I]}(z) = n - 2\alpha \geq 0$ for each $z \in V - I$, where I is an independent set in G with $|I| = \alpha$.

[2] $NK(G) \leq \left(\frac{e}{\alpha}\right)^{\alpha} \left(\frac{(\delta + \Delta)^2}{4\delta}\right)^{n - \alpha}$ with equality if and only if G is a connected regular bipartite graph with partition sets of I and V – I such that $|I| = |V - I|$, where I is an independent set in G with $|I| = \alpha$.

[3] $NK(G) \leq \left(\frac{(\delta + n - \alpha)^2}{4\delta}\right)^{\alpha} \Delta^{n - \alpha}$ with equality if and only if G is $K_{\alpha}^{c} \vee H$, where H = G[V - I], I is a maximum independent set in G with $|I| = \alpha$, $d(y) = \delta = (n - \alpha)$ for each $y \in I$, and $d(z) = \Delta$ for each $z \in V - I$.

[4] $NK(G) \leq \frac{(\delta + n - \alpha)^{2\alpha} (\delta + \Delta)^{2(n - \alpha)}}{(4\delta)^n}$ with equality if and only if G is a regular graph such that $d(x) = n - \alpha$ for each $x \in V$, $d_{G[I]}(z) = \alpha$ and $d_{G[V - I]}(z) = n - 2\alpha \geq 0$ for each $z \in V - I$, where I is an independent set in G with $|I| = \alpha$.

Theorem 1.2. Let G be a k-connected ($k \geq 2$) graph of order $n \geq 3$ vertices and size e.

[1] If $NK(G) \geq (n - k - 1)^{k + 1} \left(\frac{(\delta + \Delta)^2}{4\delta}\right)^{n - k - 1}$, then G is Hamiltonian.

[2] $NK(G) \geq \left(\frac{e}{k+1}\right)^{k+1} \left(\frac{(\delta + \Delta)^2}{4\delta}\right)^{n - k - 1}$, then G is Hamiltonian.

[3] $NK(G) \geq \left(\frac{(\delta + n - k - 1)^2}{4\delta}\right)^{k+1} \Delta^{n - k - 1}$, then G is Hamiltonian or G is $K_{k+1}^{c} \vee H$, where H = G[V - I], I is an independent set in G with $|I| = k + 1$, $|V(H)|| = k$, $d(y) = \delta = (n - k - 1)$ for each $y \in I$, and $d(z) = \Delta$ for each $z \in V - I$.

[4] $NK(G) \geq \frac{(\delta + n - k - 1)^{2(k+1)} (\delta + \Delta)^{2(n - k - 1)}}{(4\delta)^n}$, then G is Hamiltonian.

Theorem 1.3. Let G be a k-connected ($k \geq 1$) graph of order $n \geq 9$ vertices and e edges.

[1] If $NK(G) \geq (n - k - 2)^{k + 2} \left(\frac{(\delta + \Delta)^2}{4\delta}\right)^{n - k - 2}$, then G is traceable.

[2] $NK(G) \geq \left(\frac{e}{k+2}\right)^{k+2} \left(\frac{(\delta + \Delta)^2}{4\delta}\right)^{n - k - 2}$, then G is traceable.

[3] $NK(G) \geq \left(\frac{(\delta + n - k - 2)^2}{4\delta}\right)^{k+2} \Delta^{n - k - 2}$, then G is traceable or G is $K_{k+2}^{c} \vee H$, where H = G[V - I], I is an independent set in G with $|I| = k + 2$, $|V(H)| = k$, $d(y) = \delta = (n - k - 2)$ for each $y \in I$, and $d(z) = \Delta$ for each $z \in V - I$.

[4] $NK(G) \geq \frac{(\delta + n - k - 2)^{2(k+2)} (\delta + \Delta)^{2(n-k-2)}}{(4\delta)^n}$, then G is traceable.

2. Lemmas

The following results will be used as lemmas when we prove our theorems. Lemma 2.1 is Corollary 4 on Page 67 in [3]. See also [13].

Lemma 2.1. Let the real numbers γ_k (k = 1, 2, … , n) satisfy $0 < m \leq \gamma_k \leq M$. Then

$$\sum_{k=1}^{n} \frac{1}{\gamma_k} \sum_{k=1}^{n} \gamma_k \leq \frac{(m+M)^2}{4mM} n^2.$$

If M > m, then the equality sign holds in above inequality if and only if n is an even number; while, at the same time, for n/2 values of k one has $\gamma_k = m$ and for the remaining n/2 values of k one has $\gamma_k = M$. If M = m, the equality always holds in the above inequality.

Lemma 2.2 below is the well-known AM-GM inequality. See, for example, Theorem 1 on Page 27 in [9].

Lemma 2.2. Let $a_1, a_2, ..., a_n$ be positive real numbers. Then

$$(a_1 a_2 \cdots a_n)^{1/n} \leq (a_1 + a_2 + \cdots + a_n)/n$$

with equality if and only if $a_1 = a_2 = \cdots = a_n$.

The next two results are from [2].

Lemma 2.3. Let G be a k-connected graph of order $n \geq 3$. If $\alpha \leq k$, then G is Hamiltonian.

Lemma 2.4. Let G be a k-connected graph of order n. If $\alpha \leq k + 1$, then G is traceable.

Lemma 2.5 below is from [10].

Lemma 2.5. Let G be a balanced bipartite graph of order 2n with bipartition (A, B). If $d(x) + d(y) \geq n + 1$ for any $x \in A$ and any $y \in B$ with $xy \notin E$, then G is Hamiltonian.

3. Proofs

Proof of Theorem 1.1. Let I be an independent set in G with $|I| = \alpha$. Clearly, $1 \leq \alpha \leq n - 1$.

Claim 1. $\sum_{y \in I} d(y) \leq e \leq \sum_{z \in V - I} d(z)$ with either of the two equalities if and only if G is a bipartite graph with partition sets of I and V - I.

Proof of Claim 1. Note that $\sum_{y \in I} d(y) = |E(I, V - I)| \leq \sum_{z \in V - I} d(z)$ and $\sum_{z \in V - I} d(z) + \sum_{y \in I} d(y) = 2e$. Thus we have that $\sum_{y \in I} d(y) \leq e \leq \sum_{z \in V - I} d(z)$.

If $\sum_{y \in I} d(y) = e$ (resp., $\sum_{z \in V - I} d(z) = e$), then $\sum_{z \in V - I} d(z) = e$ (resp., $\sum_{y \in I} d(y) = e$). Thus G is a bipartite graph with partition sets of I and V - I.

If G is a bipartite graph with partition sets of I and V - I, it is evident that $\sum_{y \in I} d(y) = \sum_{z \in V - I} d(z) = e$. □

Claim 2. $\sum_{y \in I} d(y) \le \frac{\alpha(\delta + n - \alpha)^2}{4\delta}$ with equality if and only if $d(y) = \delta = n - \alpha$ for each $y \in I$.

Proof of Claim 2. Note that $\delta \le d(y) \le n - \alpha$ for each $y \in I$. From Lemma 2.1, we have that

$$\frac{\alpha}{n-\alpha} \sum_{y \in I} d(y) \le \sum_{y \in I} \frac{1}{d(y)} \sum_{y \in I} d(y) \le \frac{(\delta + n - \alpha)^2}{4\delta(n-\alpha)} \alpha^2.$$

Thus $\sum_{y \in I} d(y) \le \frac{\alpha(\delta + n - \alpha)^2}{4\delta}$.

If $\sum_{y \in I} d(y) = \frac{\alpha(\delta + n - \alpha)^2}{4\delta}$, then we, from Lemma 2.1 and the proofs above, have that $d(y) = \delta = n - \alpha$ for each $y \in I$.

If $d(y) = \delta = n - \alpha$ for each $y \in I$, it is easy to verify that $\sum_{y \in I} d(y) = \frac{\alpha(\delta + n - \alpha)^2}{4\delta}$. □

Claim 3. $\sum_{z \in V - I} d(z) \le \frac{((n-\alpha)(\delta + \Delta)^2}{4\delta}$ with equality if and only if $d(z) = \delta = \Delta$ for each $z \in V - I$.

Proof of Claim 3. Note that $\delta \le d(z) \le \Delta$ for each $z \in V - I$. From Lemma 2.1, we have that

$$\frac{n-\alpha}{\Delta} \sum_{z \in V - I} d(z) \le \sum_{z \in V - I} \frac{1}{d(z)} \sum_{z \in V - I} d(z) \le \frac{(\delta + \Delta)^2}{4\delta\Delta} (n - \alpha)^2.$$

Thus $\sum_{z \in V - I} d(z) \le \frac{(n-\alpha)(\delta + \Delta)^2}{4\delta}$.

If $\sum_{z \in V - I} d(z) = \frac{(n-\alpha)(\delta + \Delta)^2}{4\delta}$, then we, from Lemma 2.1 and the proofs above, have that $d(z) = \delta = \Delta$ for each $z \in V - I$.

If $d(z) = \delta = \Delta$ for each $z \in V – I$, it is easy to verify that $\sum_{z \in V - I} d(z) = \frac{(n-\alpha)(\delta + \Delta)^2}{4\delta}$. □

Proof of [1] in Theorem 1.1. Notice that $d(y) \le (n - \alpha)$ for each $y \in I$. From Lemma 2.2 and Claim 3, we have that

$$NK(G) = \prod_{x \in V} d(x) = \prod_{y \in I} d(y) \prod_{z \in V-I} d(z)$$
$$\le (n - \alpha)^{\alpha} \left(\frac{\sum_{z \in V-I} d(z)}{n - \alpha}\right)^{n - \alpha} \le (n - \alpha)^{\alpha} \left(\frac{(\delta + \Delta)^2}{4\delta}\right)^{n - \alpha}.$$

If $NK(G) = (n - \alpha)^{\alpha} \left(\frac{(\delta + \Delta)^2}{4\delta}\right)^{n - \alpha}$, we, from the above proofs, have that $d(y) = (n - \alpha)$ for each $y \in I$ and $d(z) = \delta = \Delta$ for each $z \in V - I$. Thus G is a regular graph such that $d(x) = n - \alpha$ for each $x \in V$, $d_{G[I]}(z) = \alpha$ and $d_{G[V - I]}(z) = n - 2\alpha \ge 0$ for each $z \in V - I$, where I is an independent set in G with $|I| = \alpha$.

If G is a regular graph such that d(x) = n - α for each x ∈ V, $d_{G[V - I]}(z) = n - 2\alpha \geq 0$ and $d_{G[I]}(z) = \alpha$ for each z ∈ V - I, where I is an independent set in G with |I| = α. It is easy to verify that NK(G) = $(n-\alpha)^{\alpha}\left(\frac{(\delta+\Delta)^2}{4\delta}\right)^{n-\alpha}$. □

Proof of [2] in Theorem 1.1. From Lemma 2.2, Claim 1, and Claim 3, we have that

$$NK(G) = \prod_{x \in V} d(x) = \prod_{y \in I} d(y) \prod_{z \in V-I} d(z)$$

$$\leq \left(\frac{\sum_{y \in I} d(y)}{\alpha}\right)^{\alpha} \left(\frac{\sum_{z \in V-I} d(z)}{n-\alpha}\right)^{n-\alpha} \leq \left(\frac{e}{\alpha}\right)^{\alpha} \left(\frac{(\delta+\Delta)^2}{4\delta}\right)^{n-\alpha}.$$

If NK(G) = $\left(\frac{e}{\alpha}\right)^{\alpha} \left(\frac{(\delta+\Delta)^2}{4\delta}\right)^{n-\alpha}$, we, from the above proofs, have that $\sum_{y \in I}$ d(y) = e and d(z) = δ = Δ for each z ∈ V - I. Thus G is a connected regular bipartite graph with partition sets of I and V – I such that |I| = |V – I|.

If G is a connected regular bipartite graph with partition sets of I and V - I such that |I| = |V – I|, it is easy to verify that NK(G) = $\left(\frac{e}{\alpha}\right)^{\alpha} \left(\frac{(\delta+\Delta)^2}{4\delta}\right)^{n-\alpha}$. □

Proof of [3] in Theorem 1.1. From Lemma 2.2 and Claim 2, we have that

$$NK(G) = \prod_{x \in V} d(x) = \prod_{y \in I} d(y) \prod_{z \in V-I} d(z)$$

$$\leq \left(\frac{\sum_{y \in I} d(y)}{\alpha}\right)^{\alpha} \Delta^{n-\alpha} \leq \left(\frac{(\delta+n-\alpha)^2}{4\delta}\right)^{\alpha} \Delta^{n-\alpha}.$$

K(G) = $\left(\frac{(\delta+n-\alpha)^2}{4\delta}\right)^{\alpha} \Delta^{n-\alpha}$, we, from the above proofs, have that d(y) = (n - α) = δ for each y ∈ I and d(z) = Δ for each z ∈ V - I. Thus G is $K_{\alpha}^{c} \vee H$, where H = G[V - I], I is a maximum independent set in G with |I| = α, d(y) = δ = (n - α) for each y ∈ I, and d(z) = Δ for each z ∈ V - I.

If G is $K_{\alpha}^{c} \vee H$, where H = G[V - I], I is a maximum independent set in G with |I| = α, d(y) = δ = (n - α) for each y ∈ I, and d(z) = Δ for each z ∈ V – I, it is easy to verify that

NK(G) = $\left(\frac{(\delta+n-\alpha)^2}{4\delta}\right)^{\alpha} \Delta^{n-\alpha}$. □

Proof of [4] in Theorem 1.1. From Lemma 2.2, Claim 2, and Claim 3, we have that

$$NK(G) = \prod_{x \in V} d(x) = \prod_{y \in I} d(y) \prod_{z \in V-I} d(z)$$

$$\leq \left(\frac{\sum_{y \in I} d(y)}{\alpha}\right)^{\alpha} \left(\frac{\sum_{z \in V-I} d(z)}{n-\alpha}\right)^{n-\alpha} \leq \left(\frac{(\delta+n-\alpha)^2}{4\delta}\right)^{\alpha} \left(\frac{(\delta+\Delta)^2}{4\delta}\right)^{n-\alpha}$$

$$= \frac{(\delta+n-\alpha)^{2\alpha}(\delta+\Delta)^{2(n-\alpha)}}{(4\delta)^n}.$$

If $NK(G) = \frac{(\delta + n - \alpha)^{2\alpha} (\delta + \Delta)^{2(n-\alpha)}}{(4\delta)^n}$, we, from the above proofs, have that $d(y) = \delta = n - \alpha$ for each $y \in I$ and $d(z) = \delta = \Delta$ for each $z \in V - I$. Thus G is a regular graph such that $d(x) = n - \alpha$ for each $x \in V$, $d_{G[I]}(z) = \alpha$ and $d_{G[V - I]}(z) = n - 2\alpha \geq 0$ for each $z \in V - I$, where I is an independent set in G with $|I| = \alpha$.

If G is a regular graph such that $d(x) = n - \alpha$ for each $x \in V$, $d_{G[I]}(z) = \alpha$ and $d_{G[V - I]}(z) = n - 2\alpha \geq 0$ for each $z \in V - I$, where I is an independent set in G with $|I| = \alpha$, it is easy to verify that $NK(G) = \frac{(\delta + n - \alpha)^{2\alpha} (\delta + \Delta)^{2(n-\alpha)}}{(4\delta)^n}$. □

Proof of Theorem 1.2. Let G be a k-connected graph ($k \geq 2$) with $n \geq 3$ vertices and e edges satisfying exactly one of four conditions in Theorem 1.2. Suppose G is not Hamiltonian. Then Lemma 3 implies that $\alpha \geq k + 1$. Also, we have that $n \geq 2\delta + 1$ otherwise $\delta \geq n/2$ and G is Hamiltonian. Note that we now have that $n \geq 2\delta + 1 \geq 2k + 1$. Let $I_1 := \{u_1, u_2, ..., u_\alpha\}$ be a maximum independent set in G. Then $I := \{u_1, u_2, ..., u_{k+1}\}$ is an independent set in G.

Proof of [1] in Theorem 1.2. Using proofs which are similar to the ones in Proof [1] in Theorem 1.1 and the given conditions in this case, we have that

$$(n - k - 1)^{k+1} \left(\frac{(\delta + \Delta)^2}{4\delta}\right)^{n-k-1} \leq NK(G) \leq (n - k - 1)^{k+1} \left(\frac{(\delta + \Delta)^2}{4\delta}\right)^{n-k-1}.$$

Thus $n - 2(k + 1) \geq 0$ and $\delta = n - k - 1 \geq \frac{n - k - 1 + k + 1}{2} = \frac{n}{2}$. Therefore, G is Hamiltonian, a contradiction. □

Proof of [2] in Theorem 1.2. Using proofs which are similar to the ones in Proof [2] in Theorem 1.1 and the given conditions in this case, we have that

$$\left(\frac{e}{k+1}\right)^{k+1} \left(\frac{(\delta + \Delta)^2}{4\delta}\right)^{n-k-1} \leq NK(G) \leq \left(\frac{e}{k+1}\right)^{k+1} \left(\frac{(\delta + \Delta)^2}{4\delta}\right)^{n-k-1}.$$

Thus G is a connected regular bipartite graph with partition sets of I and V - I such that $|I| = |V - I|$. Therefore Lemma 2.5 implies G is Hamiltonian, a contradiction. □

Proof of [3] in Theorem 1.2. Using proofs which are similar to the ones in Proof [3] in Theorem 1.1 and the given conditions in this case, we have that

$$\left(\frac{(\delta + n - k - 1)^2}{4\delta}\right)^{k+1} \Delta^{n-k-1} \leq NK(G) \leq \left(\frac{(\delta + n - k - 1)^2}{4\delta}\right)^{k+1} \Delta^{n-k-1}.$$

Thus $d(y) = \delta = n - k - 1$ for each $y \in I$ and $d(z) = \Delta$ for each $z \in V - I$. If $n = 2k + 1$, then G is $K_{k+1}^c \vee H$, where $H = G[V - I]$ and $|V(H)| = k$, I is an independent set in G with $|I| = k + 1$, $d(y) = \delta = (n - k - 1)$ for each $y \in I$, and $d(z) = \Delta$ for each $z \in V - I$. If $n \geq 2k + 2$, then $\delta = n - k - 1 \geq \frac{n - k - 1 + k + 1}{2} = \frac{n}{2}$ and G is Hamiltonian, a contradiction. □

Proof of [4] in Theorem 1.2. Using proofs which are similar to the ones in Proof [4] in Theorem 1.1 and the given conditions in this case, we have that

$$\frac{(\delta + n - k - 1)^{2(k+1)}\,(\delta + \Delta)^{2(n-k-1)}}{(4\delta)^n} \leq \mathrm{NK}(G) \leq \frac{(\delta + n - k - 1)^{2(k+1)}\,(\delta + \Delta)^{2(n-k-1)}}{(4\delta)^n}.$$

Thus $n - 2(k + 1) \geq 0$ and $\delta = n - k - 1 \geq \frac{n - k - 1 + k + 1}{2} = \frac{n}{2}$. Therefore, G is Hamiltonian, A contradiction. □

Recall the fact that a graph G is traceable if $\delta(G) \geq (n - 1)/2$. Using Lemma 2.4, the fact just mentioned, and the proofs which are similar to the ones in the Proof of Theorem 1.2, we can prove Theorem 1.3. The details of the proofs of Theorem 1.3 are skipped here.

4. Conclusions

Using Schweitzer's inequality, we obtain new upper bounds of the Narumi-Katayama index of a connected graph. We, using the ideas for obtaining the upper bounds, further present sufficient conditions based on the Narumi-Katayama index, minimum degree, and maximum degree for Hamiltonian and traceable graphs.

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Author's Contribution. The author solely conceived the study, conducted the research, performed the analysis, and prepared the manuscript.

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