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# Weakly Complemented Nearlattice

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***Abstract.*** This research focuses on homomorphisms on nearlattices. We first develop a comprehensive concept of homomorphism in the context of nearlattices and establish a fundamental homomorphism theorem for these structures. Building upon this framework, we derive several significant and novel results concerning homomorphic images of semiprime ideals, highlighting their structural properties and mathematical significance. We examine these results in detail and demonstrate the broader applicability and depth of the developed homomorphism theory. Furthermore, we established that a mapping $f : S \to \{\{a\}^{\perp\perp} : a \in S\}$ between semilattices qualifies as a homomorphism within a 0-distributive semilattice precisely when it fulfils the designated condition.

Finally, we presented various characterizations of weakly complemented nearlattices relative to the mapping $J$, thereby extending and enriching the existing theory.

***Keywords:*** Semiprime ideals, 0-distributive semilattice, congruence, meet homomorphism, homomorphism, weakly complemented near-lattice.

***AMS Mathematics Subject Classification (2020):*** 06A06, 06A12, 06B10

## 1. Introduction

Varlet [11] first introduced the concept of 0-distributive lattices. Since then, this notion has attracted considerable attention, and numerous authors, including [1, 3, 5, 6, 7, 8, 9], have previously investigated 0-distributivity in the settings of lattices as well as semilattices. In particular, this concept has been shown to play an important role in understanding the structural properties of nearlattices and related algebraic systems.

A nearlattice is an algebraic structure that forms a meet-semilattice and satisfies an additional condition concerning upper bounds. Specifically, whenever any two elements of the nearlattice have a common upper bound, their least upper bound (or supremum) exists in the structure. This additional condition is referred to as the upper bound property.

Thus, a nearlattice can be viewed as a meet-semilattice in which every pair of elements that has at least one common upper bound also possesses a unique least upper bound.

Figure 1

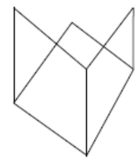

Figure 2

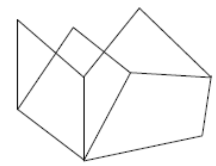

Figure 3

Let $S$ be a nearlattice whose least element is 0. If the condition $a \wedge b = 0 = a \wedge c$ implies the existence of an element $d \in S$ such that $a \wedge d = 0$ for some $d \geq b, c$, then the nearlattice $S$ is said to be 0-distributive [2]. The notion of semiprime ideals was originally introduced for lattices in [10] as a tool for studying decomposition and ideal-theoretic properties. This concept was recently expanded to meet semilattices by Noor and Begum [6]. Then Zaidur, Bazlar and Noor introduced semiprime ideal in a nearlattice [12]. Thereby broadening its applications and uses.

An ideal $J$ of a nearlattice $S$ is referred to as semiprime if for any elements $p, q, r \in S$, the inclusion of $p \wedge q \in J$, $p \wedge r \in J$, then $p \wedge t \in J$ for some $t \geq q, r$. This concept implies that if $(0]$ is a semiprime ideal of $S$, then a nearlattice $S$ with 0 is 0-distributive. Furthermore, a meet semilattice $S$ is said to be directed above if for every pair of elements $a, b \in S$, there exists an element $c \in S$ such that $c \geq a, b$. This property ensures the existence of common upper bounds and is fundamental in the study of order-theoretic structures. It is well known that all distributive and modular semilattices have the directed-above property. Also, it has been established in [2] that every 0-distributive meet semilattice is also directed-above, further highlighting the significance of 0-distributivity in semilattice theory.

Let $S$ and $T$ represent two nearlattices. A mapping $f : S \to T$ is considered a homomorphism if it retains the meet operation.

Let $S$ and $T$ be two nearlattices, and let $f$ be a mapping from $S$ into $T$. The mapping $f$ is said to be a homomorphism of nearlattices if it preserves the meet operation. In other words, for every pair of elements, the image of their meet under $f$ is equal to the meet of their images in $T$. Thus, $f(x \wedge y) = f(x) \wedge f(y)$ such that for all $x, y \in S$.

Therefore, a homomorphism preserves the fundamental meet structure of the nearlattice while mapping elements from $S$ to corresponding elements of $T$. This preservation ensures that the algebraic relationship between any two elements with respect to the meet operation remains unchanged under the mapping.

Thus, a homomorphism obeys the algebraic structure of nearlattices by maintaining the meet relationships between the elements. A homomorphism is said to be a 0-homomorphism if it preserves the least element that is, if $f(0) = 0$. Injective homomorphisms are referred to as monomorphisms or embeddings, as they faithfully embed one nearlattice into another. Surjective homomorphisms are called epimorphisms, and when $f : S \to T$ is an epimorphism, the nearlattice $T$ is said to be a homomorphic image of $S$. An epimorphism that is also injective is known as an isomorphism.

Isomorphisms preserve the entire structure of nearlattices and therefore establish a structural equivalence between them. A homomorphism from a nearlattice $S$ to itself is called an endomorphism, while an isomorphism from $S$ onto itself is referred to as an automorphism.

If there is an isomorphism $f : S \to T$, then two nearlattices $S$ and $T$ are said to be isomorphic. The nearlattices in this instance have the same algebraic structures, and this relationship is represented symbolically by $S \cong T$ .

**The nearlattices $S$ and $T$ :**

Nearlattice $S$ : A set of three arbitrary element $S = \{0, a, b\}$ ordered such that $0 < a$ and $0 < b$. Then element $a$ and $b$ are incomparable.

Nearlattice $T$ : A collection of sets $T = \{\phi, \{1\}, \{2\}\}$ ordered by set inclusion $\subseteq$. The element $\{1\}$ and $\{2\}$ are incomparable.

A mapping $S$ to $T$ is bijective homomorphism. $f(0) = \phi,\ f(a) = \{1\},\ f(b) = \{2\}.$

1. Meet preservation:
   $f(a \wedge b) = f(0) = \phi.$
   $f(a) \wedge f(b) = \{1\} \cap \{2\} = \phi.$
   $f(0 \wedge a) = f(0) = \phi.$
   $f(0) \wedge f(a) = \phi \cap \{1\} = \phi.$
2. Join preservation:
   In $S$ the pair $\{a, b\}$ has no upperbound. So $a \vee b$ is undefined. In $T$ the pair $\{1\}, \{2\}$ has no upperbound, so $\{1\} \cup \{2\}$ is not in $T$
   $f(0 \vee a) = f(a) = \{1\}.$
   $f(0) \vee f(a) = \phi \cup \{1\} = \{1\}.$

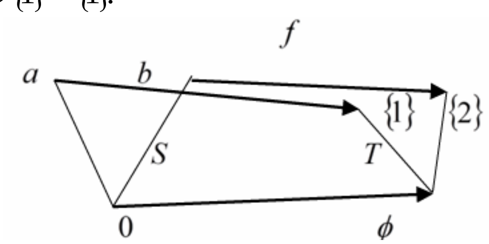


**Figure 4:**

Since $f$ is a bijective that perfectly preserves the ordered meet semilattice properties $S \cong T$ .

## 2. Semiprime ideals and homomorphism

As $J$ is an ideal of $S$, $A \subseteq S$. We define $A^{\perp_j} = \{\ x \in S : x \wedge a \in J$ for every $a \in A\ \}$, which is clearly a down set that includes $J$. Here, $A$ is related to $J$ be referred to as an annihilator of $A$. It is referred to as an annihilator ideal in relation to $J$ if it is an ideal.

According to [2, 8], $\{a\}^{\perp_j}$ is an ideal for all $a \in A$ exactly when $S$ is 0-distributive. Because of [7], the following outcome is used.

**Lemma 2.1.** Assume $J$ is an ideal in a nearlattice $S$, with $X, Y \subseteq S$ and $x, y \in S$. Then the following properties are satisfied:

(i) If $X \cap Y = J$, then $Y \subseteq X^{\perp_j}$.

(ii) $X \cap X^{\perp_j} = J$.

(iii) If $X \subseteq Y$ then $Y^{\perp_j} \subseteq X^{\perp_j}$.

(iv) If $x \leq y$ then $\{y\}^{\perp_j} \subseteq \{x\}^{\perp_j}$ and $\{x\}^{\perp_j \perp_j} \subseteq \{y\}^{\perp_j \perp_j}$.

(v) $\{x\}^{\perp_j} \cap \{x\}^{\perp_j \perp_j} = J$.

(vi) $\{x \wedge y\}^{\perp_j \perp_j} = \{x\}^{\perp_j \perp_j} \cap \{y\}^{\perp_j \perp_j}$.

(vii) $X \subseteq X^{\perp_j \perp_j}$.

(viii) $X^{\perp_j \perp_j \perp_j} = X^{\perp_j}$.

**Proposition 2.2.** Let $J$ be a semiprime ideal in a nearlattice $S$. Define a relation $R$ on $S$ by $x = y(R)$ if and only if $\{x\}^{\perp_J} = \{y\}^{\perp_J}$, then $R$ is a nearlattice congruence on $S$.

**Proof:** There is obviously an equivalency relation $R$ on $S$. Now let $x = y(R)$ and $t \in S$ Then $\{x\}^{\perp_J} = \{y\}^{\perp_J}$. Suppose $a \in \{x \wedge t\}^{\perp_J}$. Then $a \wedge x \wedge t \in J$ which implies $a \wedge t \in \{x\}^{\perp_J} = \{y\}^{\perp_J}$. Thus, $a \wedge y \wedge t \in J$ and so $a \in \{y \wedge t\}^{\perp_J}$. Therefore, z $\{x \wedge t\}^{\perp_J} \subseteq \{y \wedge t\}^{\perp_J}$. Similarly, $\{y \wedge t\}^{\perp J} \subseteq \{x \wedge t\}^{\perp J}$ and so $\{x \wedge t\}^{\perp_J} = \{y \wedge t\}^{\perp_J}$. Hence $x \wedge t \equiv y \wedge t(R)$. Now let $x \equiv y(R)$ and $x \vee t$, $y \vee t$ exist for some $t \in S$. Let $a \in \{x \vee t\}^{\perp_J}$. Then $a \wedge (x \vee t) \in J$ and so $a \wedge x, a \wedge t \in J$. This implies $a \wedge y, a \wedge t \in J$ as $\{x\}^{\perp_J} = \{y\}^{\perp_J}$. Therefore $a \wedge (y \vee t) \in J$ as $J$ is semiprime. It follows that $R$ is a nearlattice congruence on $S$.

**Note:** Let $S$ be a nearlattice, and let $\Theta$ be a congruence relation on $S$. The quotient nearlattice of $S$ with respect to $\Theta$ is denoted by $\frac{S}{\Theta}$. The elements of $\frac{S}{\Theta}$ are the congruence classes determined by $\Theta$, and hence each element of the quotient can be regarded as a subset of $S$. Suppose that the quotient nearlattice $\frac{S}{\Theta}$ possesses a zero element, denoted by $\bar{0}$. In this case, the congruence class corresponding to the zero element plays a distinguished role in the structure of $S$. We define the kernel of $\Theta$, denoted by $\ker\Theta$, to be the zero congruence class of $\frac{S}{\Theta}$; that is, $\bar{0}$.

Since the zero class is closed under the appropriate operations and satisfies the defining properties of an ideal, it follows immediately that $\ker\Theta$ is an ideal of $S$. It is important to observe that the definition does not require the original nearlattice $S$ itself to possess a zero element. The existence of a zero element is required only in the quotient nearlattice $\frac{S}{\Theta}$, Thus, even when $S$ has no least or zero element, a congruence on $S$ may still rise to a quotient possessing a zero class, which can consequently be regarded as the kernel of the congruence.

Conversely, Let $J$ be an ideal of $S$. We say that $J$ is the kernel of a homomorphism if there is a congruence $\Theta$ on $S$ where $J = \ker\Theta$.

In other words, an ideal $J$ is called a kernel precisely when it occurs as the zero congruence class of the quotient $\frac{S}{\Theta}$ for some congruence $\Theta$ on $S$. Equivalently, $J$ is a kernel if and only if it is a complete congruence class associated with a suitable congruence relation on $S$. This characterization establishes a natural connection between ideals and congruences in nearlattices such as, while every kernel obtained from a suitable congruence is an ideal, an arbitrary ideal need not necessarily arise as the kernel of a congruence. Hence, the notion of a kernel provides a way of identifying those ideals that are compatible with the congruence structure of the nearlattice.

Since for every $x \in S$, and any $i \in J$, $x \geq x \wedge i \in J$, hence $\frac{[x]}{\Theta} \geq J$ in $\frac{S}{\Theta}$, so $J$ is the zero element of $\frac{S}{\Theta}$.

An equivalence relation $\Theta$ of a nearlattice $S$ is called a congruence relation if $x_i \equiv y_i(\Theta)$ for $i = 1,2$ $(x_i, y_i \in S)$, then

(i) $x_1 \wedge x_2 \equiv y_1 \wedge y_2(\Theta)$, and

(ii) $x_1 \vee x_2 \equiv y_1 \vee y_2(\Theta)$, provided $x_1 \vee x_2$ and $y_1 \vee y_2$ exist.

**Meet homomorphism in nearlattice:**

Let $S_1$ and $S_2$ be two nearlattice a mapping $f : S_1 \to S_2$ satisfies $f(x \wedge y) = f(x) \wedge f(y)$ for all $x, y \in S_1$.

The homomorphism theorem for lattices is well known and can be found in Theorem 11 in Grätzer [4]. Since nearlattices share many structural properties with lattices, the same techniques used in the lattice case can apply with only minor modifications. Using these analogous arguments, we can formulate a corresponding homomorphism theorem for nearlattices. The proof follows the similar essential steps as same as the proof of the lattice homomorphism theorem.

**Theorem 2.3.** (A Fundamental Homomorphism Result for Nearlattice Structures).

Whenever a nearlattice $S$ is mapped homomorphically onto another structure, the resulting image can always be realized as a quotient of $S$ under an appropriate congruence relation. In particular, if $\Phi : S \to T$ is a surjective homomorphism onto $T$ and $\Theta$ is the congruence relation on $S$ defined by $x \equiv y(\Theta)$ if and only if $\Phi(x) \equiv \Phi(y)$. Then $S/_{\Theta} \cong T$.

**Proof:** Define a map $\Psi : \frac{S}{\Theta} \to T$ by $\Psi(x\Theta) = \Phi(x)$, for all $x \in S$, $x\Theta = x \wedge a = J, \forall a \in J$

**A. $\Psi$ is well defined:**

We have to show that $\Psi(x\Theta)$ is independent of choice of representative $x$ for each $x\Theta$.

Let $x\Theta = y\Theta,\ x, y \in S$. Then $\Theta \in \frac{S}{\Theta}, y \wedge a = J$ for all $a \in J$.

Then $\Psi(x\Theta) = \Psi(y \wedge a\Theta)$

$\Rightarrow \Phi(x) = \Phi(y\Theta) = \Phi(y \wedge a) = \Phi(y) \wedge \Phi(a) = \Phi(y) \wedge J = \Phi(y) = \ker\Phi$, since $J \in \frac{S}{\Theta}$.

Thus, $\Psi$ is well define.

**B. $\Psi$ is an isomorphism:**

**1. Claim $\Psi$ is a homomorphism.**

Let $x, y \in S$ then $x(\Theta), y(\Theta) \in \frac{S}{\Theta}$.

Then $\Psi(x\Theta \wedge y\Theta) = \Psi(x \wedge a \wedge y \wedge a) = \Psi(x\Theta) \wedge \Psi(y\Theta)$ for all $a \in J$

$\therefore$ $\Psi$ is a homomorphism.

**2. Claim $\Psi$ is an isomorphsm.**

(i) $\Psi$ is one one:

Suppose $\Psi(x\Theta) = \Psi(y\Theta) \Rightarrow \Phi(x) = \Phi(y) \Rightarrow \Phi(x)[\Phi(y)]^{-1} = J$

Now $\Rightarrow \Phi$ is a homomorphism.

$\Rightarrow \Phi(x \wedge y^{-1}) = \Phi(x) \wedge \Phi(y^{-1}) = \Phi(x) \wedge [\Phi(y)]^{-1} \Rightarrow x \wedge y^{-1} \in \frac{S}{\Theta} = J$

$\Rightarrow x \in y\Theta \Rightarrow x\Theta = y\Theta$

Thus $\Psi$ is an one one.

(ii) $\Psi$ is onto:

Since $\Phi$ is onto, such that $t = \Phi(x)$ maps onto this $\Phi(x)$ under $\Psi$

Thus $\Psi$ is onto map.

As a result, we conclude that

$\Psi$ is isomorphism and hence $S/_{\Theta} \cong T$.

**Theorem 2.4.** Let $f: S \to T$ be a surjective homomorphism and an $I$ ideal of $S$, satisfying $f^{-1}(f(I)) = I$. Then semiprimeness is preserved under $f$; that is, if $I$ is semiprime in $S$, then it follows that $f(I)$ is therefore semiprime in $T$.

**Proof:** Assume that $I$ is semiprime. Let $x, y, z \in T$, where $x \wedge y \in f(I)$, and $x \wedge z \in f(I)$. Then, there exists $p$, $q$ and $r \in S$ such that $x = f(p), y = f(q), z = f(r)$. Now $f(p) \wedge f(q) = f(p \wedge q) \in f(I)$ and $f(p) \wedge f(r) = f(p \wedge r) \in f(I)$. This suggests $p \wedge q, p \wedge r \in I$ given that $I$ is semiprime, $s \in S, s \geq q, r$ exist such that $p \wedge s \in I$. Let $t = f(s)$, and therefore $t = f(s) \geq f(q), f(r)$. As a result $t \geq y$ and $t \geq z$, additionally $f(p) \wedge f(s) = f(p \wedge s) \in f(I)$. Consequently, $x \wedge t \in f(I)$ and $f(I)$ is semiprime.

A nearlattice $S$ is 0-distributive when its zero ideal $(0]$ is a semiprime ideal, resulting in the following outcome is an immediate consequence of the previous theorem on the preservation of semiprime ideals under surjective homomorphism.

**Corollary 2.5.** Let $S$ and $T$ be two nearlattices with $0$, $f: S \to T$ is onto, a 0-homomorphism and $f^{-1}(0) = 0$. If $S$ is 0-distributive, then $T$ is 0-distributive.

**Example:** The example mapping $N_5$ to a two element chain. The classic non-distributive but 0-distributive lattice is a pentagonal lattice $N_5$. We will map it onto the 2-element Boolean chan which also 0-distributive.

1. **Definition of posets:**
   Nearlattice : $N_5 = \{0, a, b, c, 1\}$ with relation $0 < a < b < 1$ and $0 < c < 1$.
   Nearlattice : $T = \{0', 1'\}$ with relation $0' < 1'$.
2. **Hasse diagram:**

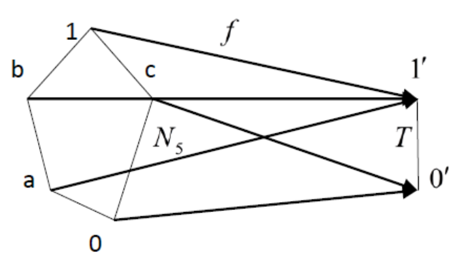


**Figure 5:**

3. **The 0-himimorpism map**

Define $f : N_5 \to T$ is a surjective (onto) mapping $f$ as follows:
$f(0) = 0', f(c) = 0', f(a) = 1', f(b) = 1', f(1) = 1'$.

**Verification of $N_5$ is 0-distributive**:

The only elements in $N_5$ whose meet is 0 and pairs involving $c$ with $a$ or $b$.
$c \wedge a = 0 \quad c \wedge b = 0$
The join $a \vee b = 0$ exists. Evaluation the condition $c \wedge (a \vee b) = c \wedge b = 0$. Since this hold for all elements. So, $N_5$ is 0-distributive.

**Verification of $f$ is a 0-homomorpism:**
Zero preserved $f(0) = 0'$ meet preserved for example $f(b \wedge c) = f(0) = 0'$ and $f(b) \wedge f(c) = 1' \wedge 0' = 0'$. Join preserved for example $f(a \vee c) = f(1) = 1'$ and $f(a) \vee f(c) = 1' \vee 0' = 1'$

**Verification of $T$ is 0-distributive:**
$T$ is a totally order chain. Every chain is completely distributive, which automatically makes $T$ is 0-distributive.

**Lemma 2.6.** Consider a nearlattice $S$, and an ideal $J$ of $S$ that satisfies the semiprime condition. $f : S \to \left\{\{x\}^{\perp_j \perp_j} : x \in S\right\}$, given by $f(x) = x^{\perp_j \perp_j}$. Consequently, the following outcomes are true:

(i) It is a meet homomorphism,

(ii) If and only if $x \in J$, then $f(x) = J$ for $x \in S$,

(iii) $f\left(\{x\}^{\perp_j}\right) = \{f(x)\}^{\perp_j}$.

**Proof:** $(i)$ Let $x, y \in S$. Now,

$$\begin{aligned} f(x \wedge y) &= \{x \wedge y\}^{\perp_j \perp_j} \\ &= \{x\}^{\perp_j \perp_j} \cap \{y\}^{\perp_j \perp_j} \\ &= f(x) \cap f(y) \\ &= f(x) \wedge f(y) \end{aligned}$$

As a result, it is a meet homomorphism.

$(ii)$ If $f(x) = J$, then $\{x\}^{\perp_j \perp_j} = J$. Thus $\{x\}^{\perp_j} = \{x\}^{\perp_j \perp_j \perp_j} = S$ and so $x \in \{x\}^{\perp_j}$.

This implies $x = x \wedge x \in J$.

Conversely, if $x \in J$, then $f(x) = \{x\}^{\perp_j \perp_j} = S^{\perp_j} = J$.

$(iii)$

$$\begin{aligned} f\left(\{x\}^{\perp_j}\right) &= \left\{\{y\}^{\perp_j \perp_j} \mid y \in \{x\}^{\perp_j}\right\} \\ f\left(\{x\}^{\perp_j}\right) &= \left\{\{y\}^{\perp_j \perp_j} \mid x \wedge y \in J\right\} \\ &= \left\{\{y\}^{\perp_j \perp_j} \mid f(x \wedge y) \in J\right\} \\ &= \left\{\{y\}^{\perp_j \perp_j} \mid f(x) \wedge f(y) \in J\right\} \\ &= \{f(x)\}^{\perp_j} \end{aligned}$$

This establishes the result conclusively.

**Corollary 2.7.** Let $S$ be defined as a 0-distributive nearlattice.
*If* $f : S \to \{\{p\}^{\perp\perp} : p \in S\}$ is defined by $f(p) = \{p\}^{\perp\perp}$, then the following results hold:

(i) $f$ is a meet homomorphism,

(ii) For $p \in S$, $f(p) = \{0\}$ if and only if $p = 0$,

(iii) $f(\{p\}^{\perp}) = \{f(p)\}^{\perp}$.

**Note:** Lemma 2.6 remains valid for any ordinary ideal $J$ of $S$. However, the assumption that $J$ is semiprime is necessary, since the sets $\{p\}^{\perp_j}$ and $\{p\}^{\perp_j\perp_j}$ form ideals only under this condition. Likewise, in a semilattice with $0$, the sets $\{p\}^{\perp}$ and $\{p\}^{\perp\perp}$ are ideals precisely when $S$ satisfies the 0-distributive property.

## 3. Weakly complemented nearlattices

Let $S$ be a nearlattice with 0. The nearlattice $S$ is said to be weakly complemented if, for any two distinct elements $a, b \in S$, there exists an element $c \in S$ such that exactly one of $a \wedge c$ and $b \wedge c$ is equal to 0.

Similarly, let $J$ be an ideal of a nearlattice $S$. We say that $S$ is weakly complemented with respect to $J$ if, for any two distinct elements $a, b \in S$, there exists an element $c \in S$ such that exactly one of $a \wedge c$ and $b \wedge c$ belongs to $J$. In particular if $a < b$, there exists an element $c \in S$ such that $a \wedge c \in J$ while $b \wedge c \notin J$. It should be noted that the following is a weakly complemented semilattice's with respect to an ideal $J$. For an ideal $J$ of a nearlattice $S$, $S$ is said to as weakly complemented in relation to $J$ if for all $x, y \in S$, $x \neq y$ implies that either $\{x\}^{\perp_j} - \{y\}^{\perp_j} \neq \Phi$, or $\{y\}^{\perp_j} - \{x\}^{\perp_j} \neq \Phi$, These semilattices are also known as disjunctive semilattices relative to $J$.

**Theorem 3.1.** Let $S$ be a nearlattice, and let $J$ be a semiprime ideal of $S$. In this situation, the following are comparable:

(i) $f : S \to \{\{x\}^{\perp_j\perp_j} \mid x \in S\}$, is defined by $f(x) = \{x\}^{\perp_j\perp_j}$ is isomorphism,

(ii) $\{x\}^{\perp_j} = \{y\}^{\perp_j} \in I(S)$, implies that $x = y$ for all $x, y \in S$,

(iii) the nearlattice $S$ is weakly complemented relative to $J$.

**Proof:** Firstly, we will prove that $(i) \Rightarrow (ii)$. Let $\{x\}^{\perp_j} = \{y\}^{\perp_j}$, and $x \neq y$. Since $f$ is an isomorphism, we have $f(x) \neq f(y)$. This implies that $\{x\}^{\perp_j\perp_j} \neq \{y\}^{\perp_j\perp_j}$. Then there exists an element $a \in \{x\}^{\perp_j\perp_j}$ such that $a \notin \{y\}^{\perp_j\perp_j}$ which implies that $a \wedge c \notin J$ for some $c \in \{y\}^{\perp_j}$,

Since $\{x\}^{\perp_j} = \{y\}^{\perp_j}$ then we have $a \wedge c \notin J$ for some $c \in \{x\}^{\perp_j}$ which implies $a \notin \{x\}^{\perp_j \perp_j}$. This gives is a contradiction. Hence $\{x\}^{\perp_j} = \{y\}^{\perp_j}$ implies $x = y$.
Secondly, we will prove $(ii) \Rightarrow (iii)$. Let $x < y$. From Lemma 2.1 and statement (ii), $\{x\}^{\perp_j} \supset \{y\}^{\perp_j}$ the conclusion follows immediately. In particular, we obtain an element $p \in \{x\}^{\perp_j}$ such that $p \notin \{y\}^{\perp_j}$ the required equality holds, which establishes that the structure $S$ is weakly complemented relative to $J$. Thirdly, we will prove that $(iii) \Rightarrow (ii)$. After that $x \neq y$, let's assume either $x \wedge y < x$ or $x \wedge y < y$. There exists $p \in \{x \wedge y\}^{\perp_j}$ such that $p \wedge x \notin J$, where $S$ is weakly complemented.
Consequently, we have $p \wedge (x \wedge y) \in J$. This suggests that $(p \wedge x) \in \{y\}^{\perp_j}$, and so $p \wedge x \notin \{x\}^{\perp_j}$. Thus, $\{x\}^{\perp_j} \neq \{y\}^{\perp_j}$, and so (ii) holds. Finally, it is needed to prove that $(ii) \Rightarrow (i)$. It is an isomorphism. For every $x, y \in S$, $x = y$

$$\begin{aligned} &\Leftrightarrow \{x\}^{\perp_j} = \{y\}^{\perp_j} \\ &\Leftrightarrow \{x\}^{\perp_j \perp_j} = \{y\}^{\perp_j \perp_j} \\ &\Leftrightarrow f(x) = f(y) \end{aligned}$$

This implies the mapping $f$ is both well-defined and injective. Its surjectivity is immediate from the definition. Moreover, Lemma 2.6 guarantees that $f$ preserves the meet "$\wedge$" operation, which implies that $f$ is an isomorphism.

## 4. Conclusion

We conclude this article by establishing a significant characterization that highlights the deep connection between 0-distributivity and semiprimeness. More precisely, we obtained the following important result, such as a nearlattice $S$ is 0-distributive if and only if $S$ is semiprime. This equivalence provides a fundamental structural criterion for identifying semiprime nearlattices. In particular, this obtained result shows that these two different algebraic conditions are actually closely related. This reveals an important properties of the ideal structure of nearlattice. The theorem also follows naturally from the preceding developments and provides a useful foundation for further investigations into the relationships among distributivity conditions, prime and semiprime ideals, and homomorphic properties of nearlattices.

**Corollary 3.2.** Assume that the nearlattice is 0-distributive. The following are therefore comparable:

(i) An isomorphism $f : S \to \{\{x\}^{\perp\perp} \mid x \in S\}$ is defined by $f(x) = \{x\}^{\perp\perp}$,

(ii) $\{x\}^{\perp} = \{y\}^{\perp} \in I(S)$, implies that $x = y$ for all $x, y \in S$,

(iii) $S$, is weakly complemented.

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