Annals of Pure and Applied Mathematics
Vol. 3, No. 1, 2013, 78-84
ISSN: 2279-087X (P), 2279-0888(online)
Published on 20 June 2013
www.researchmathsci.org

Annals of Pure and Applied Mathematics

On the Additive and Multiplicative Structure of Semirings

T. Vasanthi **and** ***N. Sulochana***

Department of Mathematics
Yogi Vemana University, Kadapa -516 003 (A. P), India
Email: vasanthi_thatimakula@yahoo.co.in; sulochananagam@gmail.com

Received 7 June 2013; accepted 17 June 2013

Abstract. Additive and multiplicative structures play an important role in determining the Structure of semiring. In this paper, we study the properties of semirings satisfying the identity $a + ab + b = a$ for all a, b in S. We characterize Boolean like semirings.

Keywords. PRD, Mono Semiring, Left (Right) Singular, Rectangular band, Zero sum free, Boolean like semiring.

AMS Mathematics Subject Classification (2010): 20M10, 16Y60

1. Introduction

A triple (S, +,·) is called a semiring if (S, +) is a semigroup; (S,·) is semigroup; a(b + c) = ab + ac and (b + c)a = ba + ca for every a, b, c in S. A semiring (S, +, ·) is said to be a totally ordered semiring if the additive semigroup (S, +) and multiplicative semigroup (S, ·) are totally ordered semigroups under the same total order relation. An element x in a totally ordered semigroup (S, ·) is non-negative (non-positive) if $x^2 \geq x$($x^2 \leq x$). A totally ordered semigroup (S, ·) is said to be non-negatively(non-positively) ordered if every one of its elements is non-negative(non-positive). (S, ·) is positively(negatively) ordered in strict sense if $xy \geq x$ and $xy \geq y$ ($xy \leq x$ and $xy \leq y$) for every x and y in S. (S, +) is said to be band if a + a = a for all a in S. A semigroup (S, +) is said to be rectangular band if a + b + a = a for all a, b in S. A semigroup (S, .) is said to be a band if $a = a^2$ for all a in S. A semigroup (S, .) is said to be left (right) singular if ab = a (ab = b) for all a, b in S. A semigroup (S, +) is said to be left (right) singular if a + b = a (a + b = b) for all a, b in S. A semiring (S, +, .) is said to be Mono semiring if a + b = ab for all a, b in S. A semiring is said to be Positive Rational Domain (PRD) if and only if (S, .) is an abelian group. A semiring (S, +, .) with additive identity zero is said to be zerosumfree semiring if x + x = 0 for all x in S.

Theorem 1.1. Let (S, +, .) be a semiring. If S contains a multiplicative identity which is also an additive identity, then (S, .) is left singular if and only if S satisfies the condition $a + ab + b = a$, for all a,b in S.

Proof: Let 'e' be the multiplicative identity which is also an additive identity
Assume that S satisfies the condition $a + ab + b = a$, for all a,b in S.

$$\Rightarrow a [e+b] + b = a \Rightarrow ab + b = a \Rightarrow [a + e] b = a \Rightarrow ab = a$$

$\therefore$ (S, .) is left singular.

Conversely, let (S, .) be a left singular semigroup

Consider $a + ab + b = a [e + b] + b = ab + b = [a + e]b = ab = a$

Hence, S satisfies the identity $a + ab + b = a$, $\forall$ a, b in S.

Theorem 1.2. Let (S, +, .) be a semiring and suppose the condition $a + ab + b = a$, for all a, b in S. If S contains a multiplicative identity which is also an additive identity then

- **(i)** $(S,+)$ is band
- **(ii)** $(S, .)$ is band
- **(iii)** $(S,+)$ is left singular
- **(iv)** $(S,+)$ is rectangular band

Proof :

(i) Assume that *S* satisfies the condition $a + ab + b = a$, for all a,b in S

Let *e* be the multiplicative identity which is also additive identity, i.e. $ae = e.a = a$. and $a + e = e + a = a$.

Let $a + ab + b = a$, for all a,b in S.

$a[e + b] + b = a \Rightarrow ab + b = a \Rightarrow a + ab + b = a + a \Rightarrow a = a + a$, for all a,b in S

$\therefore$ (S, +) is a band

(ii) Suppose $a + a^2 + a = a$ for all a in S.

$\Rightarrow a [e + a] + a = a \Rightarrow a.a + a = a \Rightarrow a^2 + a = a \Rightarrow a [a + e] = a \Rightarrow a.a = a$

$\Rightarrow a^2 = a$, for all a in S

$\therefore$ (S, .) is a band

(iii) $a + ab + b = a$

$a + [e + a] b = a$

$a + ab = a$

$a + ab + b = a + b$

$a = a + b$

$\therefore$ (S, +) is left singular Semigroup

(iv) $a + b + a = a + ab + b + b + a = a [e + b] + b + b + a = ab + b + b + a$

$= [a + e] b + b + a = ab + b + a = [a + e] b + a = ab + a = a[b + e]$

$= ab = a$ (ab = a from Theorem 1.1)

$\therefore$ (S, +) is a rectangular band

Example 1.3. This satisfies Theorem 1.2.

+	e	a	b
e	e	a	b
a	a	a	a
b	b	a	b

.	e	a	b
e	e	a	b
a	a	a	a
b	b	a	b

Definition 1.4. A semiring (S, +, .) is said to be zero square semiring if $x^2 = 0$ for all x in S, where 0 is multiplicative zero.

Theorem 1.5. Let (S, +, .) be a zero square semiring, where 0 is the additive identity. If S satisfies the identity $a + ab + b = a$ for all a, b in S, then $S^2 = \{0\}$.

Proof: Let $a + ab + b = a$ for all a, b in S.

$a(a + ab + b = a) = a.a \Rightarrow a^2 + a^2b + ab = a^2 \Rightarrow 0 + 0.b + ab = a^2 \Rightarrow a + ab = 0 \Rightarrow ab = 0.$

Also, $a + ab + b = a \Rightarrow a^2 + (ab)\, a + ba = a^2 \Rightarrow 0 + 0.a + b.a = 0 \Rightarrow 0 + ba = 0 \Rightarrow ba = 0$

$\therefore\ S^2 = \{0\}$

Example 1.6. Let S = {0, a, b}with the addition given in the table and $S^2 = \{0\}$ is an example which satisfies the conditions of theorem 1.5.

+	0	a	b
0	0	a	b
a	a	a	a
b	b	b	b

Theorem 1.7. Let (S, +, .) be a zerosumfree semiring, then

(i) $a + ab + b = a$ for all a, b in S if and only if (S, .) is right singular

(ii) If $a + ab + b = a$ then $a^2b + ab^2 = ab + (ab)^2 = 0$

Proof: (i) Consider $a + ab + b = a$ for all a, b in S

$$\Rightarrow a + ab + b + b = a + b$$
$$\Rightarrow a + ab + 0 = a + b$$
$$\Rightarrow a + a + ab = a + a + b$$
$$\Rightarrow 0 + ab = 0 + b$$
$$\Rightarrow ab = b$$

Conversely,assume, (S, .) is right singular

$$\Rightarrow ab = b$$
$$\Rightarrow a + ab = a + b$$
$$\Rightarrow a + ab + b = a + b + b$$
$$\Rightarrow a + ab + b = a + 0$$
$$\Rightarrow a + ab + b = a$$

(ii) $a^2b + ab^2 = a.ab + abb$

$= a [ab + bb]$

$= a [b + b^2]$

$= a[b + b]$ (from (i), $b.b = b$)
$= a.0$
$= 0.$

Theorem 1.8. Let $(S, +, .)$ be a semiring satisfying the identity $a + ab + b = a$ for all a, b in S and let $(S, +)$ be band then
(i) $a + b = a$, for all a, b in S.
(ii) If $(S, +)$ is commutative, then $a + ab = a$

Proof: **(i)** Consider $a + ab + b = a$ for all a, b in S
$\Rightarrow a + ab + b + b = a + b$ for all a, b in S
$\Rightarrow a + ab + b = a + b$ for all a, b in S
$\Rightarrow a = a + b$, for all a, b in S
(ii) Consider $a + ab + b = a$ for all a, b in S
$a = a + a(b + b) + b$
$= a + ab + ab + b$
$= a + ab + b + ab$
$= a + ab.$

Definition 1.9. A semiring $(S, +, \bullet)$ is said to be a Boolean semiring if $(S, \bullet)$ is a band.

Theorem 1.10. Let $(S, +, .)$ be a Boolean semiring. Then
(a) If $a + ab + b = a$ for all a, b in S, then $S = \{a, 2a\} \cup \{b, 2b\} \cup \ldots$ for all $a, b \ldots \in S$
(b) If $a + b = a$ for all a,b in S, then $a + ab + b = a$

Proof: **(a)** Let $a + a.a + a = a$ for every $a \in S$
$\Rightarrow a + a + a = a$
$\Rightarrow 3a = a$
$\Rightarrow 4a = 2a$.
This proves the theorem

(b) Consider $a + ab + b = a^2 + ab + b^2$
$= a^2 + (a + b)b$
$= a^2 + ab$
$= a(a + b)$
$= a.a$
$= a$
Hence, $a + ab + b = a$.

Example 1.11. The following are the examples of semiring satisfying Theorem 1.10(a)

(a) $S = \{a, 2a\}$

+	a	2a
a	2a	a
2a	a	2a

.	a	2a
a	a	2a
2a	2a	2a

(b) $S = \{a, 2a, b, 2b\}$

+	a	2a	b	2b
a	2a	a	2a	a
2a	a	2a	a	2a
b	2b	b	2b	b
2b	b	2b	b	2b

.	a	2a	b	2b
a	a	2a	a	2a
2a	2a	2a	2a	2a
b	b	2b	b	2b
2b	2b	2b	2b	2b

Theorem 1.12. Let $(S,+,.)$ be a semiring. If S contains a multiplicative identity which is also an absorbing element then $a + b = a$ if and only if $a + b + ab = a$ for all a, b in S.

Proof : Suppose $a + ab + b = a \Rightarrow a(1 + b) + b = a \Rightarrow a.1 + b = a \Rightarrow a + b = a$
Conversely, suppose $a + 1 = 1 \Rightarrow ab + b = b \Rightarrow a + ab + b = a + b \Rightarrow a + ab + b = a$

Definition 1.13. A C-semiring is a semiring in which

(i) $(S, +)$ is a commutative monoid
(ii) $(S, .)$ is a commutative monoid
(iii) $a.(b + c) = ab + ac$ and $(b + c).a = ba + ca$, for every a, b, c in S
(iv) $a.0 = 0.a = 0$
(v) $(S, +)$ is a band and 1 is the absorbing element of ` +'.

Theorem 1.14. Let $(S, +, \cdot)$ be a totally ordered C - semiring and satisfying the identity $a + ab + b = a$, for all a, b in S. If $(S, +)$ is p.t.o (n.t.o.), then $(S, \cdot)$ is n.t.o. (p.t.o.).

Proof: Let $a + ab + b = a$, for all a, b in S

$\Rightarrow a + a(b + b) + b = a$
$\Rightarrow a + ab + ab + b = a$
$\Rightarrow ab + a + ab + b = a$
$\Rightarrow ab + a = a$ $(\because a + ab + a = a)$ … (A)
$\Rightarrow a = ab + a \geq ab$ $(\because (S, +)$ is p.t.o.)
$\Rightarrow a \geq ab$

Suppose $ab > b$

$\Rightarrow ab + a \geq b + a$
$\Rightarrow a \geq b + a$ $(\because$ from (A))
$\Rightarrow a \geq a + b$
$\Rightarrow a + b \leq a$

which contradicts the hypothesis that $(S, +)$ is p.t.o.

$\Rightarrow ab \leq b$
$\therefore ab \leq a$ and $ab \leq b$

Hence $(S,\cdot)$ is n.t.o.
Similarly, we can prove that $(S,\cdot)$ is p.t.o if $(S, +)$ is n.t.o.

2. Boolean Like Semirings

Definition 2.1. A semigroup (S, .) is said to be weak commutative if abc = bac, for all a, b, c in S.

Definition 2.2. A non-empty set S together with two binary operations ` + ' and ` . ' satisfying the following conditions is called a Boolean like semiring, if
(i) (S, +) is a semigroup
(ii) (S, .) is a semigroup
(iii) a.(b + c) = a.b + a.c and (b + c).a = b.a + c.a
(iv) ab (a + b + ab) = ab , for all a, b in S and a.0 = 0.a = 0
(v) weak commutative

Theorem 2.3. Let (S, +, ·) be a Boolean like semiring with additive identity zero. If S is a zero square semiring, then ab = 0, for all a, b in S.

Proof: Given S is a Boolean like semiring,
We have ab (a + b + ab) = ab, for all a, b in S
$\Rightarrow$ a (ba + b^2 + bab) = ab
$\Rightarrow$ a (ba + 0 + bab) = ab (∵ S is a zero square semiring, $b^2 = 0$)
$\Rightarrow$ a (ba + abb) = ab (∵ By weak commutative)
$\Rightarrow$ a (ba + ab^2) = ab
$\Rightarrow$ a (ba + 0) = ab (∵ S is a zero square semiring, $b^2 = 0$)
$\Rightarrow$ aba = ab
$\Rightarrow$ aab = ab (∵ By weak commutative)
$\Rightarrow$ a^2b = ab
$\Rightarrow$ 0 = ab (∵ S is a zero square semiring, $a^2 = 0$)
∴ ab = 0, for all a, b in S.

Theorem 2.4. Let (S, +, ·) be a boolean like semiring with additive identity zero. If S is a zerosumfree semiring, then $a^2 = a^{2n}$ and $a^{2n+1} = a^3$ and so on , for n > 1.

Proof: Given S is a Boolean like semiring,
We have ab (a + b + ab) = ab, for all a, b in S
$\Rightarrow$ a .a (a + a + aa) = aa for all a in S
$\Rightarrow a^2 (a + a + a^2) = a^2$
$\Rightarrow a^2 (0 + a^2) = a^2$ (∵ S is a zerosumfree semiring, a + a = 0)
$\Rightarrow a^2 (a^2) = a^2 \Rightarrow a^4 = a^2$
$\Rightarrow a^4 .a = a^2.a \Rightarrow a^5 = a^3$
$\Rightarrow a^5.a = a^3.a \Rightarrow a^6 = a^4 = a^2 \Rightarrow a^6 = a^2$
$\Rightarrow a^6 .a = a^4.a = a^2.a \Rightarrow a^7 = a^5 = a^3$
i.e., $a^2 = a^4 = a^6 = a^8 = \ldots\ldots$
$\Rightarrow a^2 = a^{2n}$, for n > 1
And $a^3 = a^5 = a^7 = a^9 = \ldots\ldots$
$\Rightarrow a^3 = a^{2n+1}$, for n > 1
∴ $a^2 = a^{2n}$, for n > 1 and $a^3 = a^{2n+1}$, for n > 1

Theorem 2.5. Let (S, +, .) be a boolean like semiring. If (S, .) is a rectangular band, then a + b + ab = ab for all a, b in S. Converse is also true if (S, .) is right cancellative.

Proof: Consider ab (a +b + ab) = ab

$\Rightarrow$ a (ba + bb +bab) = ab $\Rightarrow$ a (ba + bb +b) = ab

$\Rightarrow$ (aba + abb +ab) = ab $\Rightarrow$ (a + bab + ab) = ab

$\Rightarrow$ (a + b + ab) = ab

Conversely, ab(a + b + ab) = ab

ab (ab) = ab

aba = a

REFERENCES

1. Arif Kaya and M. Satyanarayana, Semirings satisfying properties of distributive type, *Proceeding of the American Mathematical Society*, 82 (3) (1981), 341-346.
2. Jonathan S. Golan, *Semirings and their Applications*, Kluwer Academic Publishers, Dordrecht, 1999.
3. Jonathan S. Golan, Semirings and Affine Equations over Them: Theory and Applications, Kluwer Academic Publishers, 2003.
4. M.Satyanarayana, On the additive semigroup of ordered semirings, *Semigroup Forum*, 31 (1985), 193-199.
5. T. Vasanthi and N. Sulochana, Semirings satisfying the identities *International Journal of Mathematical Archive,* 3 (9), (2012), 3393-3399 .