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α-ideals in a 0-Distributive Nearlattice

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Abstract. In this paper authors proved some result on α-ideals in a 0-distributive nearlattice. They have included several characterization of these ideals. They have also given a prime Separation Theorem for α-ideals.

Keywords: 0-distributive nearlattice, α-ideals, Annihilatior ideal, Prime ideal.

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1. Introduction

α-ideals have been studied by Cornish [3] is case of distributive lattices. Recently [5] have generalized those results for distributive nearlattices. In recent years many authors e. g. [1] and [4] have studied the α-ideals in a general lattice.

[2] gives a detailed literature on nearlattice. A *nearlattice* is a meet semilattice together with the property that any two elements possessing a common upper bound have a supremum. This property is known as the *upper bound property.*

Verlet [7] have given the definition of 0-distributivity in a lattice with 0. By [8], A nearlattice S with 0 is called 0-distributive if for all $x, y, z \in S$ with $x \wedge y = 0 = x \wedge z$ and $y \vee z$ exists imply $x \wedge (y \vee z) = 0$.

We know from [8, Theorem 5] that a nearlattice S with 0 is 0-distributive if and only if $I(S)$ is pseudocomplemented and so is 0-distributive. Let L be a lattice with 0. An element a^* is called the *pseudocomplement* of a if $a \wedge a^* = 0$ and if $a \wedge x = 0$ for some $x \in L$, then $x \leq a^*$. A lattice L with 0 and 1 is called *pseudocomplemented* if its every element has a pseudocomplement. For a nearlattice S if $I(S)$ is pseudocomplemented, then for each $A \in I(S)$, A^* is also known as an annihilator ideal. An ideal I in a 0-distributive nearlattice S is called an α-ideal if for each $x \in S$, $x \in I$ implies $(x]^{**} \subseteq I$.

In this section we would like to study the α-ideals in a 0-distributive nearlattice.

For any $A \subseteq S$, we define $A^{\perp} = \{x \in S \mid x \wedge a = 0 \ \text{for all} \ a \in A\}$. $A^{\perp}$ is clearly a down set. By [8] we know that $A^{\perp}$ is an ideal if S is 0-distributive and if A is an ideal, then $A^{\perp} = A^{*}$ is the annihilator ideal.

2. Some Properties

Theorem 1. *A nearlattice S is 0-distributive if and only if*

$$((a \wedge b) \vee (a \wedge c)]^{\perp} = (a \wedge b]^{\perp} \cap (a \wedge c]^{\perp} \ \text{for all} \ a, b, c \in S.$$

Proof: Let S be 0-distributive and $x \in ((a \wedge b) \vee (a \wedge c)]^{\perp}$.

This implies $x \wedge \{(a \wedge b) \vee (a \wedge c)\} = 0$. Thus, $x \wedge (a \wedge b) = 0$, $x \wedge (a \wedge c) = 0$.

Hence, $x \in (a \wedge b]^{\perp}$ and $x \in (a \wedge c]^{\perp}$. So $x \in (a \wedge b]^{\perp} \cap (a \wedge c]^{\perp}$.

Hence $((a \wedge b) \vee (a \wedge c)]^{\perp} \subseteq (a \wedge b]^{\perp} \cap (a \wedge c]^{\perp}$.

Again, let $x \in (a \wedge b]^{\perp} \cap (a \wedge c]^{\perp}$. Then $x \wedge (a \wedge b) = 0$ and $x \wedge (a \wedge c) = 0$.

Since S is 0-distributive. So, $x \wedge \{(a \wedge b) \vee (a \wedge c)\} = 0$.

This implies $x \in ((a \wedge b) \vee (a \wedge c)]^{\perp}$.

Hence, $(a \wedge b]^{\perp} \cap (a \wedge c]^{\perp} \subseteq ((a \wedge b) \vee (a \wedge c)]^{\perp}$,

and so $((a \wedge b) \vee (a \wedge c)]^{\perp} = (a \wedge b]^{\perp} \cap (a \wedge c]^{\perp}$.

Conversely, suppose $((a \wedge b) \vee (a \wedge c)]^{\perp} = (a \wedge b]^{\perp} \cap (a \wedge c]^{\perp}$ for all $a, b, c \in S$.

Let $(a \wedge b) \wedge (a \wedge c) = 0 = (a \wedge b) \wedge (b \wedge c)$.

Then $(a \wedge b) \in (a \wedge c]^{\perp}$ and $(a \wedge b) \in (b \wedge c]^{\perp}$.

Hence $(a \wedge b) \in (a \wedge c]^{\perp} \cap (b \wedge c]^{\perp} = ((a \wedge c) \vee (b \wedge c)]^{\perp}$

Thus, $(a \wedge b) \wedge \{(a \wedge c) \vee (b \wedge c)\} = 0$ and so S is 0-distributive by [8]. ●

Theorem 2. *For any ideal I in a 0-distributive nearlattice S the set*

$$I^{e} = \left\{x \in S \,\middle|\, (a]^{*} \subseteq (x]^{*} \ \text{for some} \ a \in I\right\}$$

is the smallest α-ideal containing I and ideal I in S is an α-ideal if and only if $I = I^{e}$.

Proof: Let $x \in I^{e}$. Then $(a]^{*} \subseteq (x]^{*}$ for some $a \in I$ and so $(x]^{**} \subseteq (a]^{**}$.

Suppose $y \in (a]^{**}$. Thus $(y] \subseteq (a]^{**}$ and so $(a]^{*} \subseteq (y]^{*}$. This implies $y \in I^{e}$.

Therefore, $(a]^{**} \subseteq I^e$ and so $(x]^{**} \subseteq I^e$. It follows that I^e is an α-ideal.
Now suppose $x \in I$, Then by definition, $x \in I^e$, and so $I \subseteq I^e$.
Suppose K is an α-ideal containing I.

Let $x \in I^e$. Then $(a]^* \subseteq (x]^*$ for some $a \in I \subseteq K$. This implies $(x]^{**} \subseteq (a]^{**} \subseteq K$ as K is an α-ideal. Thus $(x] \subseteq K$ and so $x \in K$. Hence $I^e \subseteq K$. That is I^e is the smallest α-ideal containing I.●

Theorem 3. *Every annihilatior ideal in a 0-distributive nearlattice S is an α-ideal.*

Proof : Let $I = A^*$ be the annihilator ideal of S. Suppose $y \in I = A^*$.
Then $y \wedge a = 0$ for all $a \in A$. Then $(y] \wedge (a] = (0]$ and so $(y] \subseteq (a]^*$. Thus $(y]^{**} \subseteq (a]^{***} = (a]^*$ for all $a \in A$. Hence, $(y]^{**} \subseteq \bigcap_{a \in A} (a]^* = A^* = I$ and so I is an α-ideal. ●

Theorem 4. *For any ideal I in a 0-distributive nearlattice S the following are equivalent.*
(i) I *is an* α*-ideal.*
(ii) *(ii)* $I = \bigcup_{x \in I} (x]^{**}$
(iii) *For any* $x, y \in S$*, if* $x \in I$ *and* $(x]^* = (y]^*$ *then* $y \in I$.

Proof: $(i) \Rightarrow (ii)$ Let $x \in I$. Then $(x]^{**} \subseteq I$ as I is an α- ideal. So, $\bigcup_{x \in I} (x]^{**} \subseteq I$. On the other hand, for any $t \in I$. $t \in (t]^{**}$ implies $t \in \bigcup_{x \in I} (x]^{**}$.
Thus $I \subseteq \bigcup_{x \in I} (x]^{**}$, and so (ii) holds.

$(ii) \Rightarrow (iii)$ Let $x \in I$ and $(x]^* = (y]^*$. Then by (ii) $(y]^{**} = (x]^{**} \subseteq I$, and so $y \in (x]^{**} \subseteq I$.

$(iii) \Rightarrow (i)$ Let $x \in I$ and $t \in (x]^{**}$. Then $(t] \subseteq (x]^{**}$ implies $(x]^* \subseteq (t]^*$. Now choose any $r \in S$. Then $(r \wedge t] \subseteq (x]^{**}$. Again $(r \wedge t] \subseteq (t]^{**}$.
Hence $(r \wedge t] \subseteq (x]^{**} \cap (t]^{**} = (x \wedge t]^{**}$. This implies $(x \wedge t]^* \subseteq (r \wedge t]^*$.
Thus $(x \wedge t]^* = (x \wedge t]^* \cap (r \wedge t]^* = ((x \wedge t) \vee (r \wedge t)]^*$. Now $x \wedge t \in I$. So by (iii), $(x \wedge t) \vee (r \wedge t) \in I$. Then $r \wedge t \in I$ for all $r \in S$.

In particular, choose $r = t$ This implies $t \in I$. Hence $(x]^{**} \subseteq I$ and so I is an α-ideal.●

Theorem 5. *Let* S *be a 0-distributive nearlattice.* A *be a meet subsemilattice of* S. *Then* A^0 *is an* α*-ideal, where* $A^0 = \{x \in S \mid x \wedge a = 0 \text{ for some } a \in A\}$.

Proof: By [9, Theorem 5] A^0 is an ideal. Now Let $x \in A^0$ and $y \in (x]^{**}$. Clearly $x \in A^0$ implies $x \wedge a = 0$ for some $a \in A$. But then $a \in (x]^*$ and hence $y \wedge a = 0$. This shows that $y \in A^0$ consequently $(x]^{**} \subseteq A^0$. Hence A^0 is an α-ideal of S.●

Using the technique of proof of Theorem 1, we have the following result.

Corollary 6. *A nearlattice* S *with 0 is 0-distributive if and only if for all* $a,b,c \in S$ $\{(a \wedge b) \vee (a \wedge c)\}^0 = (a \wedge b)^0 \cap (a \wedge c)^0$ ●

Theorem 7. *If a prime ideal* P *of a 0-distributive nearlattice* S *is non-dense then* P *is an* α*-ideal.*

Proof: By assumption $P^* \neq (0]$. Hence there exists $x \in P^*$ such that $x \neq 0$. But then $(x]^* \supseteq P^{**}$ gives $(x]^* \supseteq P$ as $P \subseteq P^{**}$. Furthermore if $t \in (x]^*$, then $x \wedge t = 0 \in P$. But as P is a prime ideal, $t \in P$ (since $P \cap P^* = (0] \Rightarrow x \notin P$). This implies $(x]^* \subseteq P$. Combining both the inclusions, we get $P = (x]^*$. Hence P is an annihilator ideal and so by Theorem3, P is an α-ideal.●

Let S be a 0-distributive nearlattice. For an element $x \in S$, the ideals I of the form $(x]^*$ are called the annulets of S.

Corollary 8. *Every non-dense prime ideal of a 0-distributive nearlattice is an annulet.*

Proof: It is trivial from the proof of Theorem 7. ●

Lemma 9. *For an* α*-ideal* I *of a 0-distributive nearlattice* S,

$$I = \{y \in S \mid (y] \subseteq (x]^{**} \text{ for some } x \in I\}.$$

Proof: Let $a \in I$. Then $(a] \subseteq (a]^{**}$ implies that $a \in R.H.S$.

Conversely, let $a \in R.H.S$. Then $(a] \subseteq (x]^{**}$ for some $x \in I$. Since I is an α-ideal, so $(x]^{**} \subseteq I$ and so $(a] \subseteq I$. Hence $a \in I$.●

We conclude the paper with a prime Separation Theorem for α-ideals in a 0-distributive nearlattice. This result is also a generalization of the result [1, Theorem 11].

Theorem 10. *Let F be a filter and I be an α-ideal in a 0-distributive nearlattice S such that $I \cap F = \phi$. Then there exists a prime α-ideal $P \supseteq I$ such that $P \cap F = \phi$.*

Proof : Let χ be the collection of all filters containing F and disjoint from I. χ is non-empty as $F \in \chi$ Then by [6,lemma3], there exists a maximal filter Q containing F and disjoint from I. Suppose Q is not prime . Then there exist $f, g \notin Q$ such that $f \vee g$ exists and $f \vee g \in Q$. Then by [10, lemma 4], there exist $a \in Q$, $b \in Q$ such that $a \wedge f \in I$ and $b \wedge g \in I$. Thus we have $a \wedge b \wedge f \in I$ and $a \wedge b \wedge g \in I$. Then by lemma 9, $(a \wedge b \wedge f] \subseteq (x]^{**}$ and $(a \wedge b \wedge g] \subseteq (y]^{**}$ for some $x, y \in I$.
Choose any $t \in Q$. Then $(a \wedge b \wedge f] \wedge (t] \subseteq (t]^{**} \wedge (x]^{**}$.
That is $(a \wedge b \wedge t \wedge f] \subseteq (t \wedge x]^{**}$. Similarly, $(a \wedge b \wedge t \wedge g] \subseteq (t \wedge y]^{**}$. Thus we have $(a \wedge b \wedge t \wedge f] \wedge (t \wedge x]^{*} = (0] = (a \wedge b \wedge t \wedge g] \wedge (t \wedge y]^{*}$. That is
$(a \wedge b \wedge t] \wedge (t \wedge x]^{*} \wedge (t \wedge y]^{*} \wedge (f] = (0] = (a \wedge b \wedge t] \wedge (t \wedge x]^{*} \wedge (t \wedge y]^{*} \wedge (g]$.
Since $I(S)$ is 0-distributive, it follows that

$$(a \wedge b \wedge t] \wedge (t \wedge x]^{*} \wedge (t \wedge y]^{*} \wedge ((f] \vee (g]) = (0].$$

That is, $(a \wedge b \wedge t] \wedge ((t \wedge x) \vee (t \wedge y)]^{*} \wedge (f \vee g] = (0]$, $(t \wedge x) \vee (t \wedge y)$ exists by the upper bound property of S and $(t \wedge x) \vee (t \wedge y) \in I$ as $x, y \in I$. Therefore, $(a \wedge b \wedge t] \wedge (f \vee g] \subseteq ((t \wedge x) \vee (t \wedge y)]^{**}$, which implies by Lemma 9, that is $a \wedge b \wedge t \wedge (f \vee g) \in I$. But $a \in Q, b \in Q, t \in Q$, $f \vee g \in Q$ imply
$a \wedge b \wedge t \wedge (f \vee g) \in Q$ which is a contradiction to $Q \cap I = \varphi$. Therefore, Q must be prime. Thus $P = S - Q$ is a prime ideal containing I such that $P \cap Q = \varphi$.
Let $x \in P$. If $x \in I$, then $(x]^{**} \subseteq I \subseteq P$. Again if $x \in P - I$, then by maximality of Q, there exists $a \in Q$ such that $a \wedge x \in I$. Thus, $(a]^{**} \wedge (x]^{**} \subseteq I \subseteq P$. Since $(a]^{**} \not\subseteq P$, so $(x]^{**} \subseteq P$ as P is prime. Therefore P is an α-ideal. ●

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