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Coupled Coincidence Point Results in *G*-Complete Fuzzy Metric Spaces

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Abstract. Coupled coincidence and fixed point problems have begun to be considered only very recently. In this paper we work out a coupled coincidence point theorem for a compatible pair of mappings in fuzzy metric spaces which is G-complete. The space is assumed to be endowed with a partial ordering. We use a combination of analytic and order theoretic concepts in our theorem. The result is illustrated with an example.

Keyword: Fuzzy metric space, Coupled fixed point, Coupled coincidence point, Compatible mappings, G-completeness.

AMS Mathematics Subject Classification (2010): 54H25, 47H10

1. Introduction and Preliminaries

The purpose of this paper is to establish a Coupled coincidence point theorem in fuzzy metric spaces as defined by George and Veeramani [6]. The space is assumed to be G-complete. Although there are several other definitions of fuzzy metric space, fixed point theory developed in this space more elaborately. Some examples of the works are provided in [2, 5, 10, 11]. One of the reasons for this is that the topology in such space is Hausdorff. Coupled fixed point result in this fuzzy metric space was proved by Zhu [15] and was followed in works like [4, 8].

Definition 1.1. ([13]) *A binary operation* $*:[0,1]^2 \to 0,1]$ *is called a* t*-norm if the following properties are satisfied:*

(i) $*$ is associative and commutative,

(ii) $a*1=a$ for all $a\in[0,1]$,

(iii) $a*b\le c*d$ whenever $a\le c$ and $b\le d$, for all $a,b,c,d\in[0,1]$.

Generic examples of $t-$norm are $a*_1 b=\min\{a,b\}$, $a*_2 b=\dfrac{ab}{max\{a,b,\lambda\}}$ for $0<\lambda<1$, $a*_3 b=ab$, $a*_4 b=max\{a+b-1,0\}$.

The following is the definition given by George and Veeramani [6].

Definition 1.2. ([6]) *The 3-tuple* $(X, M, *)$ *is called a fuzzy metric space in the sense of George and Veeramani if* X *is a non-empty set,* $*$ *is a continuous* t*-norm and* M *is a fuzzy set on* $X^2 \times (0, \infty)$ *satisfying the following conditions for each* $x, y, z \in X$ *and* $t, s > 0$*:*

(i) $M(x, y, t) > 0$,
(ii) $M(x, y, t) = 1$ if and only if $x = y$,
(iii) $M(x, y, t) = M(y, x, t)$,
(iv) $M(x, y, t) * M(y, z, s) \leq M(x, z, t+s)$ and
(v) $M(x, y, .) : (0, \infty) \to [0,1]$ is continuous.

Let $(X, M, *)$ be a GV-fuzzy metric space. For $t > 0$, $0 < r < 1$, the open ball $B(x, r, t)$ with center $x \in X$ is defined by

$$B(x, r, t) = \{y \in X : M(x, y, t) > 1 - r\}.$$

A subset $A \subset X$ is called open if for each $x \in A$, there exist $t > 0$ and $0 < r < 1$ such that $B(x, r, t) \subset A$. Let τ denote the family of all open subsets of X. Then τ is called the topology on X induced by the fuzzy metric M. This topology is Hausdorff and first countable [6].

A metric space (X, d) can be considered as a fuzzy metric space $(X, M, *)$ with $a * b = \min\{a, b\}$ and M defined as $M(x, y, t) = \dfrac{t}{t + d(x, y)}$.

Amongst other inequivalently defined fuzzy metric spaces, we will only consider this space and hence will refer to it simply as a fuzzy metric space.

Example 1.3. ([6]) *Let* $X = \mathbb{R}$. *Let* $a * b = a.b$ *for all* $a, b \in [0. \infty)$. *For each* $t \in (0, \infty)$ *, let*

$$M(x, y, t) = e^{-\frac{|x-y|}{t}},$$

for all $x, y \in X$. Then $(X, M, *)$ is a fuzzy metric space.

Definition 1.4. ([6,14]) *Let* $(X, M, *)$ *be a fuzzy metric space.*

(i) A sequence $\{x_n\}$ in X is said to be convergent to a point $x \in X$ if $\lim_{n \to \infty} M(x_n, x, t) = 1$ for all $t > 0$.

(ii) A sequence $\{x_n\}$ in X is called a Cauchy sequence if for each $0 < \varepsilon < 1$ and $t > 0$, there exists $n_0 \in \mathbb{N}$ such that $M(x_n, x_m, t) > 1 - \varepsilon$ for each $n, m \geq n_0$.

(iii) A fuzzy metric space in which every Cauchy sequence is convergent is said to be complete.

(iv) A sequence $\{x_n\}$ in X is called a G-Cauchy sequence if for each

$0 < \varepsilon < 1$ and $t > 0$, there exists $n_0 \in \mathrm{N}$ such that $M(x_n, x_{n+p}, t) > 1 - \epsilon$ for each $n \geq n_0$ and fixed p.

(v) A fuzzy metric space in which every G-Cauchy sequence is convergent is said to be G-complete. G-completeness is weaker than completeness.

The following lemma, which was originally proved for the fuzzy metric space introduced by Kramosil et al. [9] is also true in the present case.

Lemma 1.5. ([7]) *Let $(X, M, *)$ be a fuzzy metric space. Then $M(x, y, .)$ is nondecreasing for all $x, y \in X$.*

Lemma 1.6. ([12]) *M is a continuous function on $X^2 \times (0, \infty)$.*

It is our purpose in this paper to prove a coupled coincidence point theorem for two mappings in complete fuzzy metric spaces.

Let $(X, \leq)$ be a partially ordered set and F be a self map on X. The mapping F is said to be non-decreasing if for all $x_1, x_2 \in X$, $x_1 \leq x_2$ *implies* $F(x_1) \leq F(x_2)$ and non-increasing if for all $x_1, x_2 \in X$, $x_1 \leq x_2$ implies $F(x_1) \geq F(x_2)$ [1].

Definition 1.7. ([1]) *Let $(X, \leq)$ be a partially ordered set and $F : X \times X \to X$ be a mapping. The mapping F is said to have the mixed monotone property if F is non-decreasing in its first argument and is non-increasing in its second argument, that is, if for all $x_1, x_2 \in X$, $x_1 \leq x_2$ implies $F(x_1, y) \leq F(x_2, y)$ for fixed $y \in X$ and if for all $y_1, y_2 \in X$, $y_1 \leq y_2$ implies $F(x, y_1) \geq F(x, y_2)$, for fixed $x \in X$.*

Definition 1.8. ([3]) *Let $(X, \leq)$ be a partially ordered set and $F : X \times X \to X$ and $g : X \to X$ be two mappings. The mapping F is said to have the mixed g-monotone property if F is monotone g-non-decreasing in its first argument and is monotone g-non-increasing in its second argument, that is, if for all $x_1, x_2 \in X$, $gx_1 \leq gx_2$ implies $F(x_1, y) \leq F(x_2, y)$ for all $y \in X$ and if for all $y_1, y_2 \in X$, $gy_1 \leq gy_2$ implies $F(x, y_1) \geq F(x, y_2)$, for any $x \in X$.*

Definition 1.9. ([1]) *Let X be a nonempty set. An element $(x, y) \in X \times X$ is called a coupled fixed point of the mapping $F : X \times X \to X$ if*

$$F(x, y) = x, F(y, x) = y.$$

Definition 1.10. ([3]) *Let X be a nonempty set. An element $(x, y) \in X \times X$ is called a coupled coincidence point of the mappings $F : X \times X \to X$ and $g : X \to X$ if*

$$gx = F(x, y), gy = F(y, x).$$

Definition 1.11. ([3]) *Let* (X,d) *be a metric space. The mappings* F *and* g *where* $F: X\times X \to X$ *and* $g: X \to X$ *, are said to be compatible if*

$$\lim_{n\to\infty} d(g(F(x_n, y_n)), F(g(x_n), g(y_n))) = 0$$

and

$$\lim_{n\to\infty} d(g(F(y_n, x_n)), F(g(y_n), g(x_n))) = 0,$$

whenever $\{x_n\}$ and $\{y_n\}$ are sequences in X such that $\lim_{n\to\infty} F(x_n, y_n)$ = $\lim_{n\to\infty} g(x_n) = x$ and $\lim_{n\to\infty} F(y_n, x_n)$ = $\lim_{n\to\infty} g(y_n) = y$ for some $x, y \in X$.

Definition 1.12. ([8]) *Let* $(X, M, *)$ *be a fuzzy metric space. The mappings* F *and* g *where* $F: X\times X \to X$ *and* $g: X \to X$ *, are said to be compatible if for all* $t > 0$

$$\lim_{n\to\infty} M(g(F(x_n, y_n)), F(g(x_n), g(y_n), t) = 1$$

and

$$\lim_{n\to\infty} M(g(F(y_n, x_n)), F(g(y_n), g(x_n), t) = 1,$$

whenever $\{x_n\}$ and $\{y_n\}$ are sequences in X such that $\lim_{n\to\infty} F(x_n, y_n)$ = $\lim_{n\to\infty} g(x_n) = x$ and $\lim_{n\to\infty} F(y_n, x_n)$ = $\lim_{n\to\infty} g(y_n) = y$ for some $x, y \in X$.

The following lemma is used in our theorem.

Lemma 1.13. ([14]) *Let* $(X, M, *)$ *be a G-complete fuzzy metric space. Then* $\{x_n\}$ *converges whenever* $\lim_{n\to\infty} M(x_n, x_{n+1}, t) = 1$ *for all* $t > 0$.

2. Main results

Theorem 2.1. *Let* $(X, \leq)$ *be a partially ordered set and* $(X, M, *)$ *be a G-complete fuzzy metric space where* $a*b \geq a.b$ *for all* $a, b \in [0,1]$. *Let* $F: X\times X \to X$ *and* $g: X \to X$ *be two mappings such that* F *has the mixed* g *-monotone property and that the following conditions are satisfied:*

(i) F is continuous and $F(X\times X) \subseteq gX$,

(ii) g is continuous and monotonic increasing,

(iii) (g, F) is a compatible pair,

(iv) $$M(F(x,y), F(u,v), t) + q(1 - max\{M(gx, F(u,v), t), M(gu, F(x,y), t)\}), \geq \gamma(M(gx, gu, t) * M(gy, gv, t)) \quad (2.1)$$

for all $x, y, u, v \in X$, $t > 0$ with $gx \leq gu$ and $gy \geq gv$ where $\gamma: [0,1] \to [0,1]$ is a continuous function such that $\gamma(a) > \sqrt{a}$ for each $0 < a < 1$. If there exist $x_0, y_0 \in X$ such that $gx_0 \leq F(x_0, y_0)$ and $gy_0 \geq F(y_0, x_0)$,then there exist $x, y \in X$ such that $gx = F(x,y)$ and $gy = F(y,x)$, that is, F and g have a coupled coincidence point in X.

Proof: *Let* x_0, y_0 *be two points in X be such that* $gx_0 \leq F(x_0, y_0)$ and $gy_0 \geq F(y_0, x_0)$. *We define the sequence* $\{x_n\}$ *and* $\{y_n\}$ *in X as follows:*

$gx_1 = F(x_0, y_0)$ and $gy_1 = F(y_0, x_0)$

$gx_2 = F(x_1, y_1)$ and $gy_2 = F(y_1, x_1)$

and, in general, for all $n \geq 0$,

$$gx_{n+1} = F(x_n, y_n) \text{ and } gy_{n+1} = F(y_n, x_n). \tag{2.2}$$

This construction is possible by the condition $F(X \times X) \subseteq gX$.

Next, we prove that for all $n \geq 0$,

$$gx_n \leq gx_{n+1} \tag{2.3}$$

and

$$gy_n \geq gy_{n+1}. \tag{2.4}$$

Since $gx_0 \leq F(x_0, y_0)$ and $gy_0 \geq F(y_0, x_0)$ and since $gx_1 = F(x_0, y_0)$ and $gy_1 = F(y_0, x_0)$, we have $gx_o \leq gx_1$ and $gy_0 \geq gy_1$. Therefore (2.3) and (2.4) hold for $n = 0$.

Let (2.3) and (2.4) hold for some $n = m$. As F has the mixed g-monotone property and $gx_m \leq gx_{m+1}$ and $gy_m \geq gy_{m+1}$, from (2.2), we get

$$gx_{m+1} = F(x_m, y_m) \leq F(x_{m+1}, y_m) \text{ and } F(y_{m+1}, x_m) \leq F(y_m, x_m) = gy_{m+1} \tag{2.5}$$

Also, for the same reason, we have

$$gx_{m+2} = F(x_{m+1}, y_{m+1}) \geq F(x_{m+1}, y_m) \text{ and } F(y_{m+1}, x_m) \geq F(y_{m+1}, x_{m+1}) = gy_{m+2} \tag{2.6}$$

Then from (2.5) and (2.6),

$gx_{m+1} \leq gx_{m+2}$ and $gy_{m+1} \geq gy_{m+2}$.

Then, by induction, it follows that (2.3) and (2.4) hold for all $n \geq 0$.

Let for all $t > 0, n \geq 0$,

$$\delta_n(t) = M(gx_n, gx_{n+1}, t) * M(gy_n, gy_{n+1}, t).$$

By using (2.3) and (2.4), from (2.1) and (2.2) we have for all $t > 0$ and $n \geq 1$,

$$\begin{aligned}
&M(gx_n, gx_{n+1}, t) = M(F(x_{n-1}, y_{n-1}), F(x_n, y_n), t) \\
&\geq \gamma\big(M(gx_{n-1}, gx_n, t) * M(gy_{n-1}, gy_n, t)\big) \\
&\qquad - q(1 - \max\{M(gx_{n-1}, F(x_n, y_n), t), M(gx_n, F(x_{n-1}, y_{n-1}), t)\}) \\
&= \gamma(M(gx_{n-1}, gx_n, t) * M(gy_{n-1}, gy_n, t) \\
&\qquad - q(1 - \max\{M(gx_{n-1}, gx_{n+1}, t), M(gx_n, gx_n, t)\}) \\
&= \gamma(M(gx_{n-1}, gx_n, t) * M(gy_{n-1}, gy_n, t) - q(1 - 1) \\
&= \gamma\big(\delta_{n-1}(t)\big).
\end{aligned}$$

Therefore for all $t > 0$ and $n \geq 1$

$$M(gx_n, gx_{n+1}, t) \geq \gamma(\delta_{n-1}(t)). \tag{2.7}$$

Similarly, by using (2.3) and (2.4), from (2.1) and (2.2) we have, for all $t > 0$ and $n \geq 1$.

$$M(gy_n, gy_{n+1}, t) = M(F(y_{n-1}, x_{n-1}), F(y_n, x_n), t)$$
$$\geq \gamma\big(M(gy_{n-1}, gy_n, t) * M(gx_{n-1}, gx_n, t)\big)$$
$$- q(1 - \max\{M(gy_{n-1}, F(y_n, x_n), t), M(gy_n, F(y_{n-1}, x_{n-1}), t)\})$$
$$= \gamma(M(gy_{n-1}, gy_n, t) * M(gx_{n-1}, gx_n, t)$$
$$- q(1 - \max\{M(gy_{n-1}, gy_{n+1}, t), M(gy_n, gy_n, t)\})$$
$$= \gamma(M(gy_{n-1}, gy_n, t) * M(gx_n - 1, gx_n, t) - q(1 - 1)$$
$$= \gamma\big(\delta_{n-1}(t)\big).$$

Therefore for all $t > 0$ and $n \geq 1$

$$M(gy_n, gy_{n+1}, t) \geq \gamma(\delta_{n-1}(t)). \tag{2.8}$$

From (2.7) and (2.8) we obtain for all $t > 0$ and $n \geq 1$,

$$\delta_n(t) \geq \gamma(\delta_{n-1}(t)) * \gamma(\delta_{n-1}(t)) \geq (\gamma(\delta_{n-1}(t)))^2 > \delta_{n-1}(t).$$

(by the properties of $*$ and γ).

Thus for each $t > 0$, $\{\delta_n(t); n \geq 0\}$ is an increasing sequence in $[0,1]$ and hence tends to a limit $a(t) \leq 1$. We claim that $a(t) = 1$ for all $t > 0$. If there exists $t_0 > 0$ such that $a(t_0) < 1$, then taking limit as $n \to \infty$ for $t = t_0$ in the first part of the above inequality, we get $a(t_0) \geq (\gamma(a(t_0)))^2 > a(t_0)$, which is a contradiction. Hence $a(t) = 1$ for every $t > 0$, that is, for all $t > 0$,

$$\lim_{n\to\infty} \delta_n(t) = \lim_{n\to\infty} M(gx_n, gx_{n+1}, t) * M(gy_n, gy_{n+1}, t) = 1.$$

Then above limit implies that $\lim_{n\to\infty} M(gx_n, gx_{n+1}, t) = \lim_{n\to\infty} M(gy_n, gy_{n+1}, t) = 1.$

Then by an application of the lemma 1.13 we obtain that $\{gx_n\}$ and $\{gy_n\}$ are both convergent sequences.

Then there exist $x, y \in X$ such that

$$\lim_{n\to\infty} gx_n = x \text{ and } \lim_{n\to\infty} gy_n = y.$$

Therefore, $\lim_{n\to\infty} gx_{n+1} = \lim_{n\to\infty} F(x_n, y_n) = x$, $\lim_{n\to\infty} gy_{n+1} = \lim_{n\to\infty} F(y_n, x_n) = y$.

Since, (g, F) is a compatible pair, using continuity of g and F, we have

$$\lim_{n\to\infty} g(gx_{n+1}) = gx = \lim_{n\to\infty} g(F(x_n, y_n)) = \lim_{n\to\infty} F(gx_n, gy_n) = F(x, y),$$

and $\lim_{n\to\infty} g(gy_{n+1}) = gy = \lim_{n\to\infty} g(F(y_n, x_n)) = \lim_{n\to\infty} F(gy_n, gx_n) = F(y, x)$.

This completes the proof of the theorem.

3. Conclusion

It remains to be investigated whether the result obtained in Theorem 2.1 is valid in a general complete fuzzy metric spaces. It is also to be investigated whether the continuity of F can be replaced with any other suitable conditions.

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