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Vertex Covering and Independence in Semigraph

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Abstract. Vertex covering and independence have been well-studied concepts in graph theory. These concepts have also been defined in semigraph. In this paper we consider a subsemigraph $G-v$ of a semigraph G where v is vertex of G. We prove that for one such subsemigraph $G-v$ the vertex covering number does not exceed the vertex covering number of G. For others subsemigraph $G-v$ it may exceed. We also prove some related results about independence in semigraph.

Keywords: Semigraph, Subsemigraph, Vertex covering number, Independence number

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1. Introduction

Semigraphs provide a generalization of graphs with many applications and scope for further research. There are concepts in graph theory, which have several variants in semigraph theory. As a results, many new theorems have appeared. Semigraphs have been well studied by several authors like [1]. In semigraphs, also some authors have defined parameters like domination number, Independence number.

In this article we consider two subsemigraphs of G whose vertex set is $V(G)-\{v\}$. In the first subsemigraph $G-v$, we consider those subedges of G which are obtained by removing the vertex v from every edge of G. In the second subsemigraph $G-v$ we consider those edges of G which do not contain the vertex v.

We would like to study the effect of removing a vertex from a semigraph on two parameters namely vertex covering number and independence number of a semigraph. These concepts have been defined in [3].

2. Preliminaries

Definition 2.1. Independence set [3]

A set $S \subseteq V$ in a semigraph G is an independent set if no edge is a subset of *S.*

An independent set with maximum cardinality is called a maximum independent set of G, and it is denoted as $\beta_0 - set$ of *G*.

The cardinality of a maximum independent set is called the independence number of *G*, it is denoted as $\beta_0(G)$.

Definition 2.2. Vertex Covering Set and *e* –Vertex Covering Numbers [1]

A subset *S* of $V(G)$ is called a vertex covering set, if every edge of *G* has non-empty intersection with *S*.

A vertex covering set with minimum cardinality is called $\alpha_0 - set$ of *G*.

The cardinality of a minimum vertex covering set of *G* is called the vertex covering number of *G* and it is denoted as $\alpha_0(G)$.

It is obvious to see that a subset S of $V(G)$ is a minimum vertex covering set if and only if $V(G) - S$ is a maximum independent set.

Note that the $\alpha_0(G) + \beta_0(G) = n = The\, number\, of\, Vertices\, G$.

Definition 2.3. Edge degree [1]

If v is a vertex of semigraph G.Then the edge degree $\deg_e v$ is defined to be the number of edges, which contain the vertex v.

3. Subsemigraph

3.1. Subsemigraph of type – 1

Here we considered the subsemigraph $G - v$ whose vertex set is $V(G) - \{v\}$ and the edge set is sub edges obtained by removing the vertex v from every edge of G. We call this subsemigraph of type 1.

In this section, we will consider the subsemigraph $G - v$ of type 1.

In the following lemma, we shall prove that the vertex covering number cannot decrease when a vertex v is removed from the semigraph G.

Lemma 3.1. *If G is a semigraph and v is a vertex of G, then* $\alpha_0(G) \le \alpha_0(G - v)$.

Proof: Let S be a minimum vertex covering set of $G - v$. Let E be any edge of G.
If $v \notin E$, then E is an edge of $G - v$ and since, S is a vertex covering set of $G - v$, $E \cap S \ne \phi$.

If E is an edge of G and $v \in E$ then $E' = E - v$ is an edge of $G - v$, since, S is a vertex covering set of $G - v, E' \cap S \ne \phi$.Hence $E \cap S \ne \phi$, hence S is a vertex covering set of G.

Therefore, $\alpha_0(G) \le |S| \le \alpha_0(G - v)$.

Now we shall state and prove the necessary and sufficient condition under which the vertex covering number of a semigraph does not change when a vertex is removed from the semigraph.

Theorem 3.2. *Let G be a semigraph and $v \in V(G)$, then $\alpha_0(G) = \alpha_0(G - v)$ if and only if there is a minimum vertex covering set S of G such that $v \notin S$.*

Proof: First, suppose that $\alpha_0(G) = \alpha_0(G-v)$. Let S be a minimum vertex covering set of $G-v$. By above lemma 3.1, S is a vertex covering set of G.

If S is not a minimum vertex covering set of G, then $\alpha_0(G) < |S| = \alpha_0(G-v)$ which is a contradiction. Hence S is a minimum vertex covering set of G and since $S \subset V(G) - \{v\}, v \notin \mathrm{S}$.

Conversely, let S be a minimum vertex covering set of G such that $v \notin S$. Let E' be any edge of $G-v$, then $E' = E - \{v\}$ for some edge E of G. Now $E \cap S \neq \phi$, and therefore $E' \cap S \neq \phi$, because $v \notin S$. Therefore, S is a vertex covering set of $G-v$

Thus, $\alpha_0(G-v) \le |S| = \alpha_0(G) \le \alpha_0(G-v)$.

Hence, $\alpha_0(G) = \alpha_0(G-v)$.

Corollary 3.3. *With notation as above,* $\alpha_0(G-v) > \alpha_0(G)$ *if and only if* $v \in S$ *, for every minimum vertex covering set* S *of* G.

Example 3.4. Consider the semigraph G (see Figure 1) with $V(G) = \{0,1,2,3,4,5,6\}$ and $E(G) = \{(1,0,4),(2,0,5),(3,0,6)\}$. Note that $\alpha_0(G) = 1$. However, $\alpha_0(G-0) = 3$. Thus, $\alpha_0(G-0) > \alpha_0(G)$.

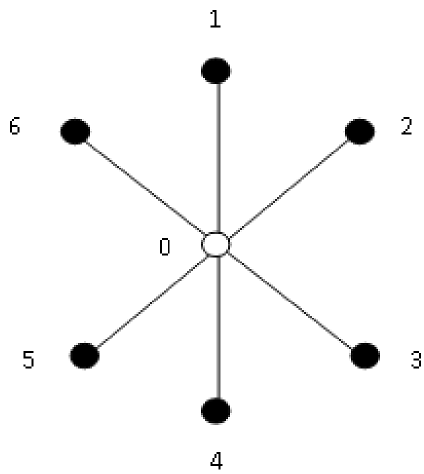

Figure 1:

Note that $\{0\}$ is the only minimum vertex covering set of G. Thus, 0 belongs to every minimum vertex covering set of G, and thus, $\alpha_0(G-0) > \alpha_0(G)$.

Here also we prove that the independence number of $G-v$ does not increase.

Theorem 3.5. *If* G *is a semigraph and* $v \in V(G)$ *then* $\beta_0(G-v) < \beta_0(G)$.

Proof: Let S be a maximum independent set of $G-v$.
We claim that $S \cup \{v\}$ is an independent set in G.
For this, suppose E is an edge of G such that $E \subset S \cup \{v\}$.

Case I: $v \notin E$
Then, $E' = E - v = E$ and E' is a subset of S. Hence S is not an independent set of $G-v$. A contradiction.
Case II: $v \in E$
Then, since $E \subset S \cup \{v\}$, $E' = E - v \subset S$ and thus, S is not an independent set of $G-v$. Again a contradiction.
So, $S \cup \{v\}$ must be an independent set of $G-v$.
Therefore, $\beta_0(G) \ge |S| + 1 > |S| = \beta_0(G-v)$. Hence, $\beta_0(G) > \beta_0(G-v)$.

Another proof of above theorem
We may note that $\alpha_0(G) + \beta_0(G) = n$. Also, $\alpha_0(G-v) + \beta_0(G-v) = n-1$.
We may note that $\alpha_0(G-v) \ge \alpha_0(G)$ and therefore $\beta_0(G-v) < \beta_0(G)$.
Now we prove necessary and sufficient condition under which $\beta_0(G-v) = \beta_0(G) - 1$.

Theorem 3.6. $\beta_0(G-v) = \beta_0(G) - 1$ *if and only if there is a maximum independent set* S *of* G *such that* $v \in S$.
Proof: Suppose there is a maximum, independent set S of G such that $v \in S$.
Now consider the set $S_1 = S - \{v\}$. First S_1 is an independent set in $G-v$. Suppose there is an edge E' of $G-v$ Such that $E' \subset S_1$. Let E be any edge of G such that $E - \{v\} = E'$. In then obviously E is a subset of S. Which contradicts the fact that S is an independent set of G.

Thus, S_1 must be an independent set in $G-v$. Since $\beta_0(G-v) < \beta_0(G)$, S_1 must be a maximum independent set of $G-v$.
Thus, $\beta_0(G-v) = |S_1| = |S| - 1 = \beta_0(G) - 1$.
Conversely, Suppose $\beta_0(G-v) = \beta_0(G) - 1$.
Let S_1 be a maximum independent set of $G-v$ and $S = S_1 \cup \{v\}$.
First, we prove that S is an independent set of G. Suppose there is an edge E of G such that $E \subset S$.

Case I: $v \notin E$
Then $E' = E - \{v\} = E$. Hence, $E' \subset S_1$. Which is contradicts the fact that S_1 is an independent set of $G-v$.

Case II: $v \in E$

Let $E' = E - \{v\}$. Since E is a subset of S, E' is a subset of S_1. Again, this contradicts the fact that S_1 is an independent set of $G - v$.

Hence, from both the cases it follows that S is an independent set of G. Also $\beta_0(G) = \beta_0(G - v) + 1$.

Therefore, S is a maximum independent set of G. Note that $v \in S$. This completes the theorem.

Remark 1. From the above theorem, it is clear that if S is a maximum independent set of G and $w \in S$, then $\beta_0(G - w) = \beta_0(G) - 1$.

Thus, if $S_1, S_2, ..., S_k$ are all maximum independent sets of G and $S = S_1 \cup S_2 \cup ... \cup S_k$ then $\beta_0(G - w) = \beta_0(G) - 1$ if and only if $w \in S$.

Thus, we have proved the following corollary.

Corollary 3.7. *The number of vertices* w *in* G *such that* $\beta_0(G - w) = \beta_0(G) - 1 = |S|$ *where* $S = S_1 \cup S_2 \cup ... \cup S_k$, *where* $\{S_1, S_2, ..., S_k\}$ *is the family of all maximum independent sets of* G.

Remark 2. From the proof of the above theorem, it is clear that if $\beta_0(G - v) = \beta_0(G) - 1$ and if S_1 is a maximum independent set of $G - v$ then $S_1 \cup \{v\}$ is a maximum independent set of G containing v.

Conversely, if S is a maximum independent set of G containing v then $S_1 = S - \{v\}$ is a maximum independent set of $G - v$.

(1) Thus, there is a one-one correspondence between the maximum independent sets of $G - v$ and the maximum independent sets of G containing the vertex v.

(2) It is also obvious that the number of maximum independent sets of G is greater than or equal to the number of maximum independent sets of $G - v$.

(3) Also it is clear that the number of maximum independent sets of G equal the number of maximum independent sets of $G - v$ if and only if v belongs to the intersection of all maximum independent sets of G.

(4) Also it may be noted that the number of vertices v such that $\beta_0(G - v) = \beta_0(G) - 1 \geq \beta_0(G)$.

Finally, we state the following corollary.

Corollary 3.8. $\beta_0(G - v) < \beta_0(G) - 1$ *if and only if* $v \notin S$ *for any maximum independent set* S *of* G.

3.2. Subsemigraph of type – 2

Now we consider a semigraph $G-v$ in which the vertex set is $V(G)-\{v\}$ and the edge set is equal to the set of those edges of G which do not contained the vertex v. This is called the subsemigraph of type – 2. In this section, we will consider subsemigraph of type –2.

First, we established that if $v \in V(G)$ then $\alpha_0(G-v) \le \alpha_0(G)$.

Lemma 3.9. *Let G be a semigraph and $v \in V(G)$ then $\alpha_0(G-v) \le \alpha_0(G)$.*

Proof: Note that every edge of $G-v$ is also and edge of G.
Let S be a minimum vertex covering set of G.

Case I: $v \in S$.
If E' is any edge of $G-v$ then E' is also an edge of G, and hence $E' \cap S \ne \phi$ since $v \notin E'$, $E' \cap (S-\{v\}) \ne \phi.$
Thus, $S-\{v\}$ is a vertex covering set of $G-v$.

Case II: $v \notin S$ **.**
Let E be any edge of $G-v$ then E is also an edge of G .Hence, $E \cap S \ne \phi$.
Thus, S is a vertex covering set of $G-v$.
Thus, S or $S-\{v\}$ is a vertex covering set of $G-v$.
Therefore, $\alpha_0(G-v) \le \alpha_0(G)$.

Lemma 3.10. *If G is a semigraph, $v \in V(G)$ and $\alpha_0(G-v) < \alpha_0(G)$ then* $\alpha_0(G-v) = \alpha_0(G)-1$.

Proof: Let S_1 be a minimum vertex covering set of $G-v$, then S_1 cannot be a vertex covering set of G. Therefore, there is an edge E of G such that $E \cap S_1 = \phi$. Then it implies that $v \in E$. Note that, $S = S_1 \cup \{v\}$ must be a vertex covering set of G. Therefore, $\alpha_0(G) = |S_1| + 1 = \alpha_0(G-v)+1$.
Thus, $\alpha_0(G-v) = \alpha_0(G)-1$.

Theorem 3.11. *Let G be a semigraph and $v \in V(G)$. Then $\alpha_0(G-v) < \alpha_0(G)$ if and only if there is a minimum vertex covering set S of G such that $v \in S$.*

Proof: Suppose $\alpha_0(G-v) < \alpha_0(G)$. Let S_1 be a minimum vertex covering set of $G-v$. Then $S = S_1 \cup \{v\}$ is minimum vertex covering set of G by lemma3.10 Thus, $v \in S$ and S is a minimum vertex covering set of G **.**

Conversely, let S be minimum vertex covering set of G such that $v \in S$.
Consider the set $S_1 = S-\{v\}$. Let E' be any edge of $G-v$. Then E' is also an edge of G. Since S is a vertex covering set of G. $E' \cap S \ne \phi$.

Since $v \notin E'$, $E' \cap (S - \{v\}) \neq \phi$. Thus $S - \{v\}$ is a vertex covering set of $G - v$
Therefore, $\alpha_0(G - v) \leq |S - \{v\}| < |S| = \alpha_0(G)$.
Hence, $\alpha_0(G - v) < \alpha_0(G)$.

Corollary 3.12. *Let G be a semigraph and $v \in V(G)$ then $\alpha_0(G - v) = \alpha_0(G)$ if and only if for every minimum vertex covering set S of G, $v \notin S$.*

Example 3.13. Let G be the semigraph whose vertex set $V(G) = \{0,1,2,3,4,5,6\}$ and $E(G) = \{(1,0,4), (2,0,5), (3,0,6)\}$. Let $v = 0$ then the subsemigraph $G - 0$ has no edges. Also, 0 belong to the only minimum vertex covering set {0}. Therefore, $\alpha_0(G - v) < \alpha_0(G)$. In fact, we may note that $\alpha_0(G) = 1$, $\alpha_0(G - 0) = 0$.
It may note that $i = 1,2,3,4,5,6$ $\alpha_0(G - i) = \alpha_0(G) = 1$.

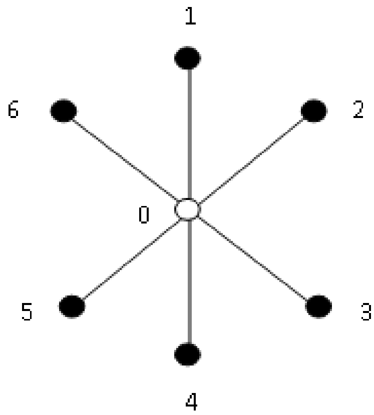

Figure 2:

Corollary 3.14. *Let G be a semigraph and $v \in V(G)$ such that v is not isolated and $\alpha_0(G - v) = \alpha_0(G)$ then for every minimum vertex covering set S of G and open neighbourhood $N(v) \cap S \neq \phi$.*

Proof: Let S be any minimum vertex covering set of G, then $v \notin S$.
Let E be any edge containing v, then $E \cap S \neq \phi$.
Thus, $N(v) \cap S \neq \phi$.

Remark 3. It may be noted that the complement of a vertex covering set is an independent set. Also, it may be noted that if $v \in V(G)$ and $v \in S$ for some minimum vertex covering set S, then $\alpha_0(G - v) < \alpha_0(G)$. Thus, we define the following two notations, $V_{cr}^{-} = \{v \in V(G) : \alpha_0(G - v) < \alpha_0(G)\}$ and
$V_{cr}^{0} = \{v \in V(G) : \alpha_0(G - v) = \alpha_0(G)\}$ then

$V_{cr}^{-} = \bigcup\{S : S\ is\ a\ \min imum\ vertex\ \text{cov}ering\ set\ of\ G\}$.

Thus,

$V_{cr}^{0} = V(G) - \bigcup\{S : S\ is\ a\ \min imum\ vertex\ \text{cov}ering\ set\ of\ G\}$

$\quad = \bigcap\{V(G) - S : S\ is\ a\ \min imum\ vertex\ \text{cov}ering\ set\ of\ G\}$

Since, the complement of every vertex covering set is an independent set it follows that V_{cr}^{0} is also is an independent set.

Thus, we have the following corollary.

Corollary 3.15. *V_{cr}^{0} is an independent set.*

Since every edge containing the vertex v intersect every minimum vertex covering set G, it followsthat $|V_{cr}^{-}| \geq$ the minimum edge degree of a semigraph G . Provided there is vertex v in the semigraph G, such that $v \in V_{cr}^{0}$. If there is no vertex in V_{cr}^{0} then every vertex is in $v \in V_{cr}^{-}$ and therefore, $|V_{cr}^{-}|$ greater than or equal to the minimum edge degree of a semigraph *G.*

Corollary 3.16. *Suppose G is a semigraph and for this semigraph $G, V_{cr}^{0} \neq \phi$ then $\alpha_0(G) \geq$ the $\min imum\ edge\ \deg ree\ of\ a\ semigraph\ G$. ($i.e. \alpha_0(G) \geq \deg_e v$).*

Proof: Let S be a minimum vertex covering set of G and $v \in V_{cr}^{0}$.

By corollary 3.15if E is an edge of G containing v, then $E \cap S \neq \phi$. Also note that if E_1 and E_2 are distinct edges of G containing v, then $E_1 \cap S \neq E_2 \cap S$ (because G is a semigraph and no two edges can intersect in more than one vertex).

Thus, $|S| \geq the\ number\ of\ edges\ containing\ vertex\ v$.

That is, $\alpha_0(G) \geq \deg_e v$.

We may prove as in the case of vertex covering that the independence number of a semigraph does not increase when a vertex is removed from the semigraph G.

Lemma 3.17. *If G is a semigraph and $v \in V(G)$ then $\beta_0(G - v) \leq \beta_0(G)$.*

Proof: Let S be a maximum independence set of $G - v$. Then obviously S is also an independent set of G .Thus, $\beta_0(G - v) \leq |S| \leq \beta_0(G)$.

Hence, $\beta_0(G - v) \leq \beta_0(G)$.

Example 3.18. Consider the semigraph G whose vertex set $V(G) = \{0,1,2,3,4,5,6\}$ and edge set $E(G) = \{(1,2),(2,3),(3,4),(4,5),(5,6),(6,1),(1,0,4,),(2,0,5),(3,0,6)\}$. In this semigraph the set $\{0,1,3,5,\}$ & $\{0,2,4,6\}$ are maximum independent sets and its independence number $\beta_0(G) = 4$. Now, let $v = 0$. The subsemigraph $G - v$ has the vertex set is $V(G-0)=\{1,2,3,4,5,6,\}$ and the edge set is $E(G-0)=\{(1,2,),(2,3),(3,4)(4,5),(5,6),(6,1)\}$. The set $\{1,3,5\}$ & $\{2,4,6\}$ are maximum independent set of $G-v$ and its independence number is $\beta_0(G-0)=3$. Note that $\beta_0(G)>\beta_0(G-0)$.

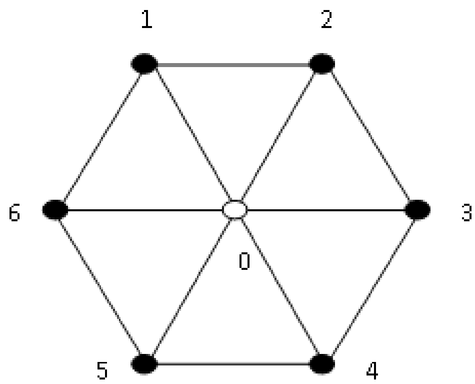

Figure 3:

Now we prove the necessary and sufficient condition under which the independence number of semigraph G does not change.

Theorem 3.15. $\beta_0(G-v)=\beta_0(G)$ *if and only if there is a maximum independent set* S *of* G *such that* $v\notin S$.

Proof: Suppose $\beta_0(G-v)=\beta_0(G)$. Let S be a maximum independent set of $G-v$. Now *S* is also an independent set of *G*.

Also, $\beta_0(G)=|S|$ and thus, *S* is a maximum independent set of G not containing the vertex v.

Conversely, let S be a maximum independent set of G such that $v\notin S$. Since $v\notin S, S$ is also an independent set of $G-v$ and therefore, $|S|\le\beta_0(G-v)$.

Thus, $\beta_0(G)\le\beta_0(G-v)\le\beta_0(G)$.

Therefore, $\beta_0(G)=\beta_0(G-v)$.

Corollary 3.19. $\beta_0(G-v)<\beta_0(G)$ *if and only if v belongs to every maximum independent set of G.*

Now we introduce the following notations,

$I^0=\{v\in V(G):\beta_0(G-v)=\beta_0(G)\}$

$I^-=\{v\in V(G):\beta_0(G-v)<\beta_0(G)\}$

Thus, from the corollary 3.19we can deduce that $I^- = \bigcap\{S : S\ is\ a\ \max imum\ independent\ set\ of\ G\}$. Also it may be noted that $I^0 = V(G) - \bigcup\{S : S\ is\ a\ \max imum\ indpendent\ set\ G\}$.

Now we prove that when α_0 decreases β_0 remains same and when α_0 remains same β_0 decreases.(Whena vertex removed from a semigraph).

Theorem 3.20. *(1) If* $\alpha_0(G-v) < \alpha_0(G)$ *then* $\beta_0(G-v) = \beta_0(G)$.

(2) If $\alpha_0(G-v) = \alpha_0(G)$ *then* $\beta_0(G-v) < \beta_0(G)$.

Proof: (1) $\alpha_0(G) + \beta_0(G) = n = number\ of\ vertices\ of\ G$.

Now, $\alpha_0(G-v) + \beta_0(G-v) = n-1$.

Thus, $\alpha_0(G) - 1 + \beta_0(G-v) = n-1$.

Hence, $\alpha_0(G) + \beta_0(G-v) = n$.

From this, it follows that, $\beta_0(G-v) = \beta_0(G)$.

Proof of (2) is similar.

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