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# A Subclass of Harmonic Functions Associated with a Convolution Structure

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***Abstract.*** In this paper, we introduce a certain subclass of harmonic univalent functions associated with convolution. We also derive the coefficient conditions, extreme points, distortion bounds, convolution conditions and convex combination for this class.

***Keywords***: Harmonic, univalent, starlike, convex, convolution

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## 1. Introduction

In 1984, Clunie and Sheil-Small [3] investigated the class $S_H$, consisting of complex valued harmonic sense-preserving univalent functions $f = h + \overline{g}$ in a simply connected domain $D \subseteq C$ defined on the open unit disc $\Delta = \{z : |z| < 1\}$ and normalized by $f(0) = f_z(0) - 1 = 0$ with h and g given by,

$$h(z) = z + \sum_{n=2}^{\infty} a_n z^n; \qquad g(z) = \sum_{n=1}^{\infty} b_n z^n, \quad |b_1| < 1. \tag{1.1}$$

Clunie and Sheil-Small [3] also considered the geometric subclasses of $S_H$ and obtained some coefficient bounds. Since then, there have been several related papers on $S_H$ and its subclasses. For a brief history on Harmonic univalent functions, see (Duren [5], Ahuja [1], Jahangiri et al. [8] and Ponnusamy and Rasila [10, 11]).

In 1999, Jahangiri [7] defined the class $T_H(\alpha)$ consisting of functions $f = h + \overline{g}$ such that h and g are of the form

$$h(z) = z - \sum_{n=2}^{\infty} |a_n| z^n; \qquad g(z) = \sum_{n=1}^{\infty} |b_n| z^n \tag{1.2}$$

which satisfy the condition

$$\frac{\partial}{\partial\theta}(\arg\ f(re^{i\theta})) \geq \alpha, \quad 0 \leq \alpha < 1, \quad |z| = r < 1.$$

In this paper, motivated by the works of Frasin [6], Rosihan Ali et al. [2], Dixit and Porwal [4], Murugusundaramoorthy et al. [9], we consider the subclass $S_H^*(\phi, \psi, \lambda, \gamma, k)$ of functions of the form (1.1) satisfying the condition,

$$\operatorname{Re}\left[\frac{(1+ke^{i\alpha})[z(h*\phi)'(z)-\overline{z(g*\psi)'(z)}]}{z'[(1-\lambda)z+\lambda[(h*\phi)(z)+\overline{(g*\psi)(z)}]]}-ke^{i\alpha}\right]\geq\gamma \tag{1.3}$$

for all real $\alpha$, $\phi(z)=z+\sum_{n=2}^{\infty}\lambda_n z^n$, $\psi(z)=z+\sum_{n=2}^{\infty}\mu_n z^n$ are analytic with conditions, $\lambda_n \geq 0$, $\mu_n \geq 0$, $0 \leq \lambda \leq 1$, $z'=\frac{\partial}{\partial\theta}(z=re^{i\theta})$, $0 \leq r < 1$, $0 \leq \theta < 2\pi$ and $0 \leq \gamma < 1$. The operator * denotes the Hadamard product or convolution of two power series.

We further let $\overline{S_H^*}(\phi,\psi,\lambda,\gamma,k)$ denote the subclass of $S_H^*(\phi,\psi,\lambda,\gamma,k)$ consisting of functions $f=h+\overline{g}\in S_H$ such that h and g are of the form (1.2).

**Remark 1.1.**

When $\alpha = 0$, $\overline{S_H^*}\left(\frac{z}{1-z},\frac{z}{1-z},1,\gamma,1\right)=T_H\left(\frac{1+\gamma}{2}\right)$ [7]

In the present paper, we obtain the coefficient inequality, extreme points, distortion bounds, convolution conditions and convex combination for the functions of the class $\overline{S_H^*}(\phi,\psi,\lambda,\gamma,k)$.

## 2. Coefficient Bounds

We begin with a sufficient condition for functions in $S_H^*(\phi,\psi,\lambda,\gamma,k)$.

**Theorem 2.1.** Let the function $f=h+\overline{g}$ be so that h and g are given by (1.1). Furthermore, let

$$\sum_{n=2}^{\infty}\left(\frac{[n(1+k)-\lambda(k+\gamma)]\lambda_n}{1-\gamma}\right)|a_n|+\sum_{n=1}^{\infty}\left(\frac{[n(1+k)+\lambda(k+\gamma)]\mu_n}{1-\gamma}\right)|b_n|\leq 1 \tag{2.1}$$

where $0 \leq \gamma < 1$, $0 \leq \lambda \leq 1$, $k \geq 0$, $\alpha$ real and if

$$n(1-\gamma)\leq[n(1+k)-\lambda(k+\gamma)]\lambda_n \leq [n(1+k)+\lambda(k+\gamma)]\mu_n .$$

Then f is sense preserving, harmonic univalent mapping in $\Delta$ and for $\lambda=\frac{1-\gamma}{1+\gamma}$, $f\in S_H^*(\phi,\psi,\lambda,\gamma,k)$.

**Proof.** First we note that f is locally univalent and sense-preserving in $\Delta$. This is because

$$\begin{aligned}|h'(z)| &\geq 1-\sum_{n=2}^{\infty}n|a_n|r^{n-1} > 1-\sum_{n=2}^{\infty}n|a_n| \\ &\geq 1-\sum_{n=2}^{\infty}\frac{[n(1+k)-\lambda(k+\gamma)]}{1-\gamma}\lambda_n|a_n| \\ &\geq \sum_{n=1}^{\infty}\frac{[n(1+k)+\lambda(k+\gamma)]}{1-\gamma}\mu_n|b_n| \\ &\geq \sum_{n=1}^{\infty}n|b_n| > \sum_{n=1}^{\infty}n|b_n|r^{n-1}\geq|g'(z)|.\end{aligned}$$

To show that f is univalent in $\Delta$, we show that $f(z_1) \neq f(z_2)$ whenever $z_1 \neq z_2$. Suppose $z_1, z_2 \in \Delta$ so that $z_1 \neq z_2$. Since the unit disc $\Delta$ is simply connected and convex, we have $z(t) = (1-t)z_1 + tz_2$ in D, where $0 \leq t \leq 1$. Then we write

$$f(z_2) - f(z_1) = \int_0^1 [(z_2 - z_1)h'(z(t)) + \overline{(z_2 - z_1)g'(z(t))}]dt$$

On dividing throughout by $z_2 - z_1 \neq 0$ and taking only the real parts we obtain

$$\operatorname{Re}\frac{f(z_2) - f(z_1)}{z_2 - z_1} = \int_0^1 \operatorname{Re}\left[h'(z(t)) + \frac{\overline{(z_2 - z_1)}}{z_2 - z_1}\overline{g'(z(t))}\right]dt$$

$$> \int_0^1 [\operatorname{Re} h'(z(t)) - |g'(z(t))|]\, dt \qquad (2.2)$$

On the other hand

$$\operatorname{Re} h'(z) - |g'(z)| \geq \operatorname{Re} h'(z) - \sum_{n=1}^{\infty} n|b_n|$$

$$\geq 1 - \sum_{n=2}^{\infty} n|a_n| - \sum_{n=1}^{\infty} n|b_n|$$

$$\geq 1 - \sum_{n=2}^{\infty} \frac{[n(1+k) - \lambda(k+\gamma)]}{1-\gamma}\lambda_n |a_n|$$

$$- \sum_{n=1}^{\infty} \frac{[n(1+k) + \lambda(k+\gamma)]}{1-\gamma}\mu_n |b_n|$$

$$\geq 0 \qquad \text{by (2.1).}$$

Therefore this together with inequality (2.2) implies the univalence of f(z).

Next we show that $f \in S_H^*(\phi, \psi, \lambda, \gamma, k)$.

From (1.3)

$$\operatorname{Re}\left\{\frac{(1+ke^{i\alpha})[z(h*\phi)'(z) - \overline{z(g*\psi)'(z)}] - ke^{i\alpha}[z'[(1-\lambda)z + \lambda[(h*\phi)(z) + \overline{(g*\psi)(z)}]]]}{z'[(1-\lambda)z + \lambda[(h*\phi)(z) + \overline{(g*\psi)(z)}]]}\right\} = \operatorname{Re}\left(\frac{A(z)}{B(z)}\right) \geq \gamma.$$

where

$$A(z) = (1+ke^{i\alpha})[z(h*\phi)'(z) - \overline{z(g*\psi)'(z)}] - ke^{i\alpha}[z'[(1-\lambda)z + \lambda[(h*\phi)(z) + \overline{(g*\psi)(z)}]]]$$

and $B(z) = z'[(1-\lambda)z + \lambda[(h*\phi)(z) + \overline{(g*\psi)(z)}]]$.

Using the fact that $\operatorname{Re} w \geq \gamma$ if and only if $|1-\gamma+w| \geq |1+\gamma-w|$ it suffices to show that

$$|A(z)+(1-\gamma)B(z)| - |(1+\gamma)B(z)-A(z)| \geq 0 \qquad (2.3)$$

substituting A(z) and B(z) in L.H.S of (2.3) and making use of (2.1) we obtain

$$\begin{aligned}
&|A(z)+(1-\gamma)B(z)|-|A(z)-(1+\gamma)B(z)|\\
&=\Big|(1+ke^{i\alpha})[z(h*\varphi)'(z)-\overline{z(g*\psi)'(z)}]-ke^{i\alpha}[z'[(1-\lambda)z+\lambda[(h*\varphi)(z)+\overline{(g*\psi)(z)}]]]\\
&\qquad+(1-\gamma)[z'[(1-\lambda)z+\lambda[(h*\varphi)(z)+\overline{(g*\psi)(z)}]]]\Big|\\
&\quad-\Big|(1+ke^{i\alpha})[z(h*\varphi)'(z)-\overline{z(g*\psi)'(z)}]-ke^{i\alpha}[z'[(1-\lambda)z+\lambda[(h*\varphi)(z)+\overline{(g*\psi)(z)}]]]\\
&\qquad-(1+\gamma)[z'[(1-\lambda)z+\lambda[(h*\varphi)(z)+\overline{(g*\psi)(z)}]]]\Big|\\
&=\Big|(2-\gamma)z+\sum_{n=2}^{\infty}[(n+\lambda(1-\gamma))+ke^{i\alpha}(n-\lambda)]\lambda_n a_n z^n-\sum_{n=1}^{\infty}[(n-(1-\gamma)\lambda)+ke^{i\alpha}(n+\lambda)]\mu_n\overline{b_n}\bar{z}^n\Big|\\
&\qquad-\Big|-\gamma z+\sum_{n=2}^{\infty}[(n-(1+\gamma)\lambda)+ke^{i\alpha}(n-\lambda)]\lambda_n a_n z^n-\sum_{n=1}^{\infty}[(n+(1+\gamma)\lambda)+ke^{i\alpha}(n+\lambda)]\mu_n\overline{b_n}\bar{z}^n\Big|\\
&\geq(2-\gamma)|z|-\sum_{n=2}^{\infty}|(n+\lambda(1-\gamma))+ke^{i\alpha}(n-\lambda)|\lambda_n|a_n||z|^n-\sum_{n=1}^{\infty}|(n-(1-\gamma)\lambda)+ke^{i\alpha}(n+\lambda)|\mu_n|b_n||z|^n\\
&\qquad-\gamma|z|-\sum_{n=2}^{\infty}|(n-(1+\gamma)\lambda)+ke^{i\alpha}(n-\lambda)|\lambda_n|a_n||z|^n-\sum_{n=1}^{\infty}|(n+(1+\gamma)\lambda)+ke^{i\alpha}(n+\lambda)|\mu_n|b_n||z|^n\\
&\geq 2(1-\gamma)|z|-2\sum_{n=2}^{\infty}[n(1+k)-\lambda(\gamma+k)]\lambda_n|a_n||z|^n-2\sum_{n=1}^{\infty}[n(1+k)+\lambda(\gamma+k)]\mu_n|b_n||z|^n\\
&\geq 2(1-\gamma)|z|\left\{\begin{aligned}&1-\sum_{n=2}^{\infty}\frac{[n(1+k)-\lambda(\gamma+k)]}{1-\gamma}\lambda_n|a_n||z|^{n-1}\\&-\sum_{n=1}^{\infty}\frac{[n(1+k)+\lambda(\gamma+k)]}{1-\gamma}\mu_n|b_n||z|^{n-1}\end{aligned}\right\}\\
&\geq 2(1-\gamma)|z|\left\{\begin{aligned}&1-\sum_{n=2}^{\infty}\frac{[n(1+k)-\lambda(\gamma+k)]}{1-\gamma}\lambda_n|a_n|\\&-\sum_{n=1}^{\infty}\frac{[n(1+k)+\lambda(\gamma+k)]}{1-\gamma}\mu_n|b_n|\end{aligned}\right\}\geq 0
\end{aligned}$$

The coefficient bound (2.1) is sharp for the function

$$f(z)=z+\sum_{n=2}^{\infty}\frac{[n(1+k)-\lambda(\gamma+k)]}{1-\gamma}\lambda_n x_n z^n+\sum_{n=1}^{\infty}\frac{[n(1+k)+\lambda(\gamma+k)]}{1-\gamma}\mu_n\overline{y_n}\bar{z}^n \quad (2.4)$$

where $\sum_{n=2}^{\infty}|x_n|+\sum_{n=1}^{\infty}|y_n|=1.$ □

Next we show that the above sufficient condition is also necessary for functions in $\overline{S_H^*}(\phi,\psi,\lambda,\gamma,k)$.

**Theorem 2.2.** Let the function $f=h+\bar{g}$ be so that h and g are given by (1.2). Then $f\in\overline{S_H^*}(\phi,\psi,\lambda,\gamma,k)$ if and only if

$$\sum_{n=2}^{\infty}\frac{[n(1+k)-\lambda(\gamma+k)]}{1-\gamma}\lambda_n|a_n|+\sum_{n=1}^{\infty}\frac{[n(1+k)+\lambda(\gamma+k)]}{1-\gamma}\mu_n|b_n|\le 1 \tag{2.5}$$

where $0\le\gamma<1$, $0\le\lambda\le 1$, $k\ge 0$, $\alpha$ real and

$$n(1-\gamma)\le[n(1+k)-\lambda(k+\gamma)]\lambda_n\le[n(1+k)+\lambda(k+\gamma)]\mu_n$$

and $|b_n|>|a_n|$, for every $n\ge 2$.

**Proof.** Since $\overline{S_H^*}(\phi,\psi,\lambda,\gamma,k)\subset S_H^*(\phi,\psi,\lambda,\gamma,k)$, we need only to prove the only if part of the theorem. For functions f(z) of the form (1.2), we note that the condition

$$\operatorname{Re}\left\{\frac{(1+ke^{i\alpha})[z(h*\phi)'(z)-\overline{z(g*\psi)'(z)}]}{z'[(1-\lambda)z+\lambda[(h*\phi)(z)+\overline{(g*\psi)(z)}]]}-ke^{i\alpha}\right\}\ge\gamma$$

is equivalent to

$$\operatorname{Re}\left\{\frac{(1+ke^{i\alpha})[z(h*\phi)'(z)-\overline{z(g*\psi)'(z)}]-ke^{i\alpha}[z'[(1-\lambda)z+\lambda[(h*\phi)(z)+\overline{(g*\psi)(z)}]]]}{z'[(1-\lambda)z+\lambda[(h*\phi)(z)+\overline{(g*\psi)(z)}]]}\right\}\ge\gamma,$$

or

$$\operatorname{Re}\left\{\frac{(1-\gamma)z-\sum_{n=2}^{\infty}[(n-\lambda\gamma)+ke^{i\alpha}(n-\lambda)]\lambda_n|a_n||z|^n-\sum_{n=1}^{\infty}[(n+\lambda\gamma)+ke^{i\alpha}(n+\lambda)]\mu_n|b_n||z|^n}{z-\sum_{n=2}^{\infty}\lambda\lambda_n|a_n||z|^n+\sum_{n=1}^{\infty}\lambda\mu_n|b_n||z|^n}\right\}\ge 0 \tag{2.6}$$

The above required condition (2.6) must hold for all real values of $\alpha$, $|z|=r<1$.
Upon choosing $\alpha=0$ and the value of z on the positive real axis where $0\le z=r<1$, we must have for $|b_n|\ge|a_n|$, for every $n\ge 2$

$$\left\{\frac{(1-\gamma)-\sum_{n=2}^{\infty}[n(1+k)-\lambda(\gamma+k)]\lambda_n|a_n|r^{n-1}-\sum_{n=1}^{\infty}[n(1+k))+\lambda(\gamma+k)]\mu_n|b_n|r^{n-1}}{1-\sum_{n=2}^{\infty}\lambda\lambda_n|a_n|r^{n-1}+\sum_{n=1}^{\infty}\lambda\mu_n|b_n|r^{n-1}}\right\}\ge 0 \tag{2.7}$$

If the condition (2.5) does not hold, then the numerator in (2.7) is negative for r sufficiently close to 1. Hence there exist $z_0 = r_0$ in (0, 1) for which the quotient in (2.7) is negative. This contradicts the required condition for $f \in \overline{S_H^*}(\phi, \psi, \lambda, \gamma, k)$ and so the proof is complete. □

## 3. Distortion Bounds

In next theorem, we determine distortion bounds for functions in $\overline{S_H^*}(\phi, \psi, \lambda, \gamma, k)$.

**Theorem 3.1.** Let $f \in \overline{S_H^*}(\phi, \psi, \lambda, \gamma, k)$ and

$$[2(1+k)-\lambda(k+\gamma)]\lambda_2 \le [n(1+k)-\lambda(k+\gamma)]\lambda_n \le [n(1+k)+\lambda(k+\gamma)]\mu_n$$

for $n \ge 2$. Then we have,

$$|f(z)| \le (1+|b_1|)r + \left(\frac{1-\gamma}{[2(1+k)-\lambda(k+\gamma)]\lambda_2} - \frac{[1+k+\lambda(k+\gamma)]\mu_1}{[2(1+k)-\lambda(k+\gamma)]\lambda_2}|b_1|\right) r^2, \quad |z| = r < 1 \tag{3.1}$$

and

$$|f(z)| \ge (1-|b_1|)r - \left(\frac{1-\gamma}{[2(1+k)-\lambda(k+\gamma)]\lambda_2} - \frac{[1+k+\lambda(k+\gamma)]\mu_1}{[2(1+k)-\lambda(k+\gamma)]\lambda_2}|b_1|\right) r^2, \quad |z| = r < 1 \tag{3.2}$$

**Proof.** We only prove the first inequality. The proof for the second inequality is similar. Let $f \in \overline{S_H^*}(\phi, \psi, \lambda, \gamma, k)$. Taking the absolute value of f we obtain

$$|f(z)| \le (1+|b_1|)r + \sum_{n=2}^{\infty}(|a_n|+|b_n|)r^n$$

$$\le (1+|b_1|)r + \sum_{n=2}^{\infty}(|a_n|+|b_n|)r^2$$

$$= (1+|b_1|)r + \frac{1-\gamma}{[2(1+k)-\lambda(k+\gamma)]\lambda_2} \sum_{n=2}^{\infty}\left(\frac{[2(1+k)-\lambda(k+\gamma)]\lambda_2}{1-\gamma}|a_n| + \frac{[2(1+k)-\lambda(k+\gamma)]\lambda_2}{1-\gamma}|b_n|\right) r^2$$

$$\le (1+|b_1|)r + \frac{1-\gamma}{[2(1+k)-\lambda(k+\gamma)]\lambda_2} \sum_{n=2}^{\infty}\left(\frac{[n(1+k)-\lambda(k+\gamma)]\lambda_n}{1-\gamma}|a_n| + \frac{[n(1+k)+\lambda(k+\gamma)]\mu_n}{1-\gamma}|b_n|\right) r^2$$

$$\le (1+|b_1|)r + \frac{1-\gamma}{[2(1+k)-\lambda(k+\gamma)]\lambda_2}\left(1 - \frac{[1+k+\lambda(k+\gamma)]\mu_1}{1-\gamma}|b_1|\right) r^2,$$

[by (2.5)]

$$\le (1+|b_1|)\, r + \left( \frac{1-\gamma}{[2(1+k)-\lambda(k+\gamma)]\lambda_2} - \frac{[1+k+\lambda(k+\gamma)]\mu_1}{[2(1+k)-\lambda(k+\gamma)]\lambda_2} |b_1| \right) r^2.$$

**4. Extreme Points**

Now we obtain the extreme points of $\overline{S_H^*}(\phi, \psi, \lambda, \gamma, k)$.

**Theorem 4.1.** Let $h_1(z) = z$,

$$h_n(z) = z - \frac{(1-\gamma)}{[n(1+k)-\lambda(k+\gamma)]\lambda_n} z^n, \ n \ge 2 \text{ and}$$

$$g_n(z) = z + \frac{(1-\gamma)}{[n(1+k)+\lambda(k+\gamma)]\mu_n} \overline{z}^n, \ n \ge 1.$$

Then $f \in \overline{S_H^*}(\phi, \psi, \lambda, \gamma, k)$ if and only if it can be expressed as

$$f(z) = \sum_{n=1}^{\infty} (x_n h_n + y_n g_n), \tag{4.1}$$

where $x_n \ge 0$, $y_n \ge 0$, $\sum_{n=1}^{\infty} (x_n + y_n) = 1$. In particular, the extreme points of $\overline{S_H^*}(\phi, \psi, \lambda, \gamma, k)$ are $\{h_n\}$ and $\{g_n\}$.

**Proof.** Suppose

$$\begin{aligned} f(z) &= \sum_{n=1}^{\infty} (x_n h_n + y_n g_n) \\ &= \sum_{n=1}^{\infty} (x_n + y_n) z - \sum_{n=2}^{\infty} \frac{(1-\gamma)}{[n(1+k)-\lambda(k+\gamma)]\lambda_n} x_n z^n + \sum_{n=1}^{\infty} \frac{(1-\gamma)}{[n(1+k)+\lambda(k+\gamma)]\mu_n} y_n \overline{z}^n \\ &= z - \sum_{n=2}^{\infty} \frac{(1-\gamma)}{[n(1+k)-\lambda(k+\gamma)]\lambda_n} x_n z^n + \sum_{n=1}^{\infty} \frac{(1-\gamma)}{[n(1+k)+\lambda(k+\gamma)]\mu_n} y_n \overline{z}^n \end{aligned}$$

Then from (2.5) we have

$$\begin{aligned} &\sum_{n=2}^{\infty} \frac{[n(1+k)-\lambda(\gamma+k)]}{1-\gamma} \lambda_n |a_n| + \sum_{n=1}^{\infty} \frac{[n(1+k)+\lambda(\gamma+k)]}{1-\gamma} \mu_n |b_n| \\ &\quad = \sum_{n=2}^{\infty} \frac{[n(1+k)-\lambda(k+\gamma)]\lambda_n}{(1-\gamma)} \left( \frac{(1-\gamma)}{[n(1+k)-\lambda(k+\gamma)]\lambda_n} x_n \right) \\ &\quad\quad + \sum_{n=1}^{\infty} \frac{[n(1+k)+\lambda(k+\gamma)]\mu_n}{(1-\gamma)} \left( \frac{(1-\gamma)}{[n(1+k)+\lambda(k+\gamma)]\mu_n} y_n \right) \\ &\quad = \sum_{n=2}^{\infty} x_n + \sum_{n=1}^{\infty} y_n = 1 - x_1 \le 1 \end{aligned}$$

and so $f \in \overline{S_H^*}(\phi, \psi, \lambda, \gamma, k)$.

Conversely, if $f \in \overline{S_H^*}(\phi, \psi, \lambda, \gamma, k)$, then

$$|a_n| \le \frac{(1-\gamma)}{[n(1+k)-\lambda(k+\gamma)]\lambda_n}$$

and

$$|b_n| \le \frac{(1-\gamma)}{[n(1+k)+\lambda(k+\gamma)]\mu_n}.$$

Setting

$$x_n = \frac{[n(1+k)-\lambda(k+\gamma)]\lambda_n}{(1-\gamma)}|a_n|, \ n \ge 2 \text{ and } y_n = \frac{[n(1+k)+\lambda(k+\gamma)]\mu_n}{(1-\gamma)}|b_n|, \ n \ge 1.$$

Then note that by Theorem 2.2, $0 \le x_n \le 1$, $(n \ge 2)$ and $0 \le y_n \le 1$, $(n \ge 1)$. We define $x_1 = 1 - \sum_{n=2}^{\infty} x_n - \sum_{n=1}^{\infty} y_n \ge 0$, by Theorem 2.2.

Consequently, we can see that f(z) can be expressed in the form (4.1). This completes the proof of the Theorem 4.1. □

## 5. Convolution and Convex Combination

In this section, we show that the class $\overline{S_H^*}(\phi, \psi, \lambda, \gamma, k)$ is invariant under convolution and convex combination of its members. The convolution of two harmonic functions

$$f(z) = z - \sum_{n=2}^{\infty} a_n z^n + \sum_{n=1}^{\infty} b_n \bar{z}^n$$

and

$$F(z) = z - \sum_{n=2}^{\infty} A_n z^n + \sum_{n=1}^{\infty} B_n \bar{z}^n,$$

is defined as

$$(f * F)(z) = f(z) * F(z) = z - \sum_{n=2}^{\infty} a_n A_n z^n + \sum_{n=1}^{\infty} b_n B_n \bar{z}^n.$$

**Theorem 5.1.** If $f \in \overline{S_H^*}(\phi, \psi, \lambda, \gamma, k)$ and $F \in \overline{S_H^*}(\phi, \psi, \lambda, \gamma, k)$ then

$$f * F \in \overline{S_H^*}(\phi, \psi, \lambda, \gamma, k).$$

**Proof.** Let $f(z) = z - \sum_{n=2}^{\infty} |a_n| z^n + \sum_{n=1}^{\infty} |b_n| \bar{z}^n$ and $F(z) = z - \sum_{n=2}^{\infty} |A_n| z^n + \sum_{n=1}^{\infty} |B_n| \bar{z}^n$ be in $\overline{S_H^*}(\phi, \psi, \lambda, \gamma, k)$, then by Theorem 2.2 we have

$$\sum_{n=2}^{\infty} \frac{[n(1+k)-\lambda(k+\gamma)]\lambda_n}{(1-\gamma)}|a_n| + \sum_{n=1}^{\infty} \frac{[n(1+k)+\lambda(k+\gamma)]\mu_n}{(1-\gamma)}|b_n| \le 1 \tag{5.1}$$

and

$$\sum_{n=2}^{\infty}\frac{[n(1+k)-\lambda(k+\gamma)]\lambda_n}{(1-\gamma)}|A_n|+\sum_{n=1}^{\infty}\frac{[n(1+k)+\lambda(k+\gamma)]\mu_n}{(1-\gamma)}|B_n|\le 1. \quad (5.2)$$

So for the coefficients of f * F we can write

$$\sum_{n=2}^{\infty}\frac{[n(1+k)-\lambda(k+\gamma)]\lambda_n}{(1-\gamma)}|a_nA_n|+\sum_{n=1}^{\infty}\frac{[n(1+k)+\lambda(k+\gamma)]\mu_n}{(1-\gamma)}|b_nB_n|$$

$$\le\sum_{n=2}^{\infty}\frac{[n(1+k)-\lambda(k+\gamma)]\lambda_n}{(1-\gamma)}|a_n|+\sum_{n=1}^{\infty}\frac{[n(1+k)+\lambda(k+\gamma)]\mu_n}{(1-\gamma)}|b_n|$$

$$\le 1.$$

Thus f * F $\in \overline{S_H^*}(\phi,\psi,\lambda,\gamma,k)$.

Finally, we prove that $\overline{S_H^*}(\phi,\psi,\lambda,\gamma,k)$ is closed under convex combinations of its members.

**Theorem 5.2.** The class $\overline{S_H^*}(\phi,\psi,\lambda,\gamma,k)$ where $f_i$ is given by

$$f_i(z)=z-\sum_{n=2}^{\infty}|a_{n,i}|z^n+\sum_{n=1}^{\infty}|b_{n,i}|\overline{z}^n. \quad (5.3)$$

is closed under convex combination.

**Proof.** For i = 1, 2, ..., suppose $f_i(z)\in\overline{S_H^*}(\phi,\psi,\lambda,\gamma,k)$ where $f_i$ is given by (5.3). Then by (2.5),

$$\sum_{n=2}^{\infty}\frac{[n(1+k)-\lambda(k+\gamma)]\lambda_n}{(1-\gamma)}|a_{n,i}|+\sum_{n=1}^{\infty}\frac{[n(1+k)+\lambda(k+\gamma)]\mu_n}{(1-\gamma)}|b_{n,i}|\le 1. \quad (5.4)$$

For $\sum_{i=1}^{\infty}t_i=1$, $0\le t_i\le 1$, the convex combination of $f_i(z)$ may be written as

$$\sum_{i=1}^{\infty}t_if_i(z)=z-\sum_{n=2}^{\infty}\left(\sum_{i=1}^{\infty}t_i|a_{n,i}|\right)z^n+\sum_{n=1}^{\infty}\left(\sum_{i=1}^{\infty}t_i|b_{n,i}|\right)\overline{z}^n.$$

Then by (5.4), we obtain

$$\sum_{n=2}^{\infty}\left[\frac{[n(1+k)-\lambda(k+\gamma)]\lambda_n}{1-\gamma}\left|\sum_{i=1}^{\infty}t_i|a_{n,i}|\right|+\sum_{n=1}^{\infty}\left[\frac{[n(1+k)+\lambda(k+\gamma)]\mu_n}{1-\gamma}\left|\sum_{i=1}^{\infty}t_i|b_{n,i}|\right|\right]\right.$$

$$=\sum_{i=1}^{\infty}t_i\left\{\sum_{n=2}^{\infty}\frac{[n(1+k)-\lambda(k+\gamma)]\lambda_n}{1-\gamma}|a_{n,i}|+\sum_{n=1}^{\infty}\frac{[n(1+k)+\lambda(k+\gamma)]\mu_n}{1-\gamma}|b_{n,i}|\right\}$$

$$\le\sum_{i=1}^{\infty}t_i=1$$

and so by Theorem 2.2, we have $\sum_{i=1}^{\infty}t_if_i(z)\in\overline{S_H^*}(\phi,\psi,\lambda,\gamma,k)$.

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