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Study on Exterior Algebra Bundle and Differential Forms

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Abstract. The main purpose of this paper is to study the applications of exterior algebra to develop the theory of differential forms, fibre bundles and connections on fibre bundles.

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1. Introduction

Exterior algebra [5] and differential forms are two important sections in differential geometry. A differential form of order r or an r-form is totally antisymmetric tensor of type $(0, r)$. At any point p on a manifold, a r-form gives a multi-linear map from the r-th exterior power of the tangent space at p to $\mathbb{R}$. A fibre bundle is a space which looks locally like a product space. It may have a different global topological structure in that the space as a whole may not be homomorphic to a product space. Using this we have developed some important theorems. In the calculus of differential forms, the local field quantities are associated with the geometric and topological property of the manifold. In this paper, we finally discuss the applications of exterior algebra bundle and differential forms on manifolds in differential geometry, generalized Stock's theorem, Laplace equation, and Maxwell's equations in $\mathbb{R}^4$.

2. Connections on vector bundles

A manifold is a topological space [8] which looks locally like $\mathbb{R}^m$, but not necessarily so globally. By introducing a chart we give a local Euclidean structure to a manifold, which enables us to use the conversional calculus of several bundles. A fibre bundle is, so to speak, a topological space which looks locally like a direct product of two topological spaces. Many theories in physics such as general relativity and gauge theories are described naturally in terms of fibre bundles.
A connection on a fiber bundle [7] is a device that defines a notion of parallel transport on the bundle; that is, a way to "connect" or identify fibers over nearby points. If the fiber bundle is a vector bundle, then the notion of parallel transport is required to be linear. Such a connection is equivalently specified by a covariant derivative, which is an

operator that can differentiate sections of that bundle along tangent directions in the base manifold.

Definition 1. A connection on a vector bundle E is a map

$$D : (E) \to (T(M) \otimes E\ ),$$

which satisfies the following conditions?

i) For any $s_1, s_2\ \in (E)$,

$$D(s_1 + s_2) = Ds_1 + Ds_2$$

ii) For $s\ \in (E)$ and any $\alpha \epsilon C^{\infty}(M)$,

$$D(\alpha s) = d\alpha\ \otimes s +\ \alpha\ Ds)$$

Suppose X is a smooth tangent vector fields on M and $s\ \in (E)$. Let $D_X\ s = < X, Ds >$ where $<,>$ represents the pairing between $T(M)$ and $T^*(M)$. Then $D_X s$ is a section of E , called the absolute differential quotient or the covariant derivative of the section s along X.

Theorem 1. [4] A connection always exists on vector bundles.

Theorem 2. [1] Suppose *X* and *Y* are two arbitrary smooth tangent vector fields on the manifold *M*. Then

$$R(X,Y) = D_X D_Y - D_Y D_X - D_{[X,Y]}$$

here *D* is the connection on a vector bundle *E* of rank *q* [1].

3. Wedge product

The wedge product of a *k*-form α and an *l*-form β is a (*k* + *l*)-form denoted $\alpha\Lambda\beta$. For example, if *k* = *l* = 1, then $\alpha\Lambda\beta$ is the 2-form whose value at a point *p* is the alternating bilinear form defined by

$$(\alpha \wedge \beta)_p(v,w) = \alpha_p(v)\beta_p(w) - \alpha_p(w)\beta_p(v)$$

for *v*, *w* ∈ T_pM. The wedge product is bilinear: for instance, if α, β, and γ are any differential forms, then

$$\alpha \wedge (\beta + \gamma) = \alpha \wedge \beta + \alpha \wedge \gamma$$

It is skew commutative meaning that it satisfies a variant of anticommutativity that depends on the degrees of the forms: if α is a *k*-form and β is an *l*-form, then $\alpha \wedge \beta = (-1)^{kl}\beta \wedge \alpha$

(i) (Distributive Law): $(\omega_1 + \omega_2) \wedge \eta = \omega_1 \wedge \eta + \omega_2 \wedge \eta), \omega \wedge (\eta_1 + \eta_2) = \omega \wedge \eta_1 + \omega \wedge \eta_2)$

(ii) (Associative Law): $(f\omega) \wedge \eta = \omega \wedge (f\eta) = f(\omega \wedge \eta)$

(iii) (Skew Symmetry): $\eta \wedge \omega = -\omega \wedge \eta$ here $\omega, \eta, \omega_1, \omega_2, \eta_1,$ and η_2 are 1-form and f is a function.

(iv) $g^*(\alpha \wedge \beta) = g^*\alpha \wedge g^*\beta,\ g^*(f\omega) = (g^*f)(g^*\omega)$ here f is a scalar function and hence can be regarded as a 0-form. Its pull back g^*f by g is just the composite $f \circ g$.

(v) (Exterior Differentiation): If $\omega = f_1 dg_1 + f_2 dg_2 + \cdots + f_m dg_m$ then we have

$$d\omega = df_1 \wedge dg_1 + df_2 \wedge dg_2 + \cdots + df_m \wedge dg_m$$

(vi) ($d^2 = 0$). For each function f, $d^2 f \equiv d(df) = 0$.

(vii) (Product Rule): $$d(f\omega) = df \wedge \omega + fd\omega$$

(viii) $$f^*(d\omega) = d(f^*\omega)$$

(ix) Let f be differentiable map, ϕ, ψ are 1-Form, then

$$d(\phi \wedge \psi) = d\phi \wedge \psi - \phi \wedge d\psi$$

We can express the 2-form $dx \wedge dy$ in Cartesian coordinates and in polar coordinates is given by

$$dx \wedge dy = rdr \wedge d\theta \qquad \text{where } x = rcos\theta,\ y = rsin\theta$$

Theorem 3. [2] If ω is a 2-form on $\mathbb{R}^{\mathbf{3}}$ such that $d\omega = 0$ then there exists a one form ξ such that $d\xi = \omega$.

Theorem 4. Suppose $f: V \to W$ is a linear map. Then f^* commutes with the exterior product, that is, for any $\varphi \in \wedge^r (W^*)$ and $\psi \in \wedge^s (W^*)$,

$$f^*(\varphi \wedge \psi) = f^*\varphi \wedge f^*\psi.$$

Proof. Choose any $v_1, \dots, v_{r+s} \in V$. Then

$$\begin{aligned} f^*(\varphi \wedge \psi)(v_1, \dots, v_{r+s}) &= \varphi \wedge \psi\, (f(v_1), \dots, f(v_{r+s})) \\ &= \frac{1}{(r+s)!} \sum_{\sigma \in S(r+s)} sgn\sigma \,.\, \varphi\left(f\left(v_{\sigma(1)}\right), \dots, f\left(v_{\sigma(r)}\right)\right). \\ &\quad \psi(f\left(v_{\sigma(r+1)}\right), \dots, f\left(v_{\sigma(r+s)}\right)) \\ &= \frac{1}{(r+s)!} \sum_{\sigma \in S(r+s)} sgn\sigma \,.\, f^*\varphi\left(v_{\sigma(1)}, \dots, v_{\sigma(r)}\right). \\ &\quad f^*\psi(v_{\sigma(r+1)}, \dots, v_{\sigma(r+s)}) \\ &= f^*\varphi \wedge f^*\psi(v_1, \dots, v_{r+s}). \end{aligned}$$

Therefore $\quad f^*(\varphi \wedge \psi) = f^*\varphi \wedge f^*\psi.$

This completes the proof of the theorem □

Definition 2. Suppose M is an m-dimensional smooth manifold. Then there exists a unique map $d: A(M) \to A(M)$ such that $d(A^r(M)) \subset A^{r+1}(M)$ and such that d satisfies the following:

1) For any ω_1 , $\omega_2 \in A(M)$, $d(\omega_1 + \omega_2) = d\omega_1 + d\omega_2$.
2) Suppose ω_1 is an exterior differential r-form. Then
$$d(\omega_1 \Lambda\, \omega_2) = d\omega_1 \Lambda\, \omega_2 + (-1)^r\ \omega_1 \Lambda\, d\omega_2.$$
3) If f is a smooth function on M, i.e., $f \in A^0(M)$, then df is precisely the differential of f.
4) If $f \in A^0(M)$, then $d(df) = 0$.

The map d defined above is called the exterior derivative.

Definition 3. The k-fold exterior product of V is a vector space $\wedge^k(V)$, together with a linear map

$$\theta: V^k = \underbrace{V \times \dots \times V}_{k\ times} \to \wedge^k(V)$$

The exterior algebra $\wedge(V) = \bigoplus_k \wedge^k(V)$ is a graded algebra, with product given by the wedge $\wedge$. For finite dimensional vector spaces it is possible to find an explicit basis for each $\wedge^k(V)$, if $e_1, \dots, e_n$ is a basis of V, then the set $\{\{e_{i_1} \wedge \dots \wedge e_{i_k}\}\ 1 \le i_1 < \cdots < i_k \le n$ is a basis of $\wedge^k(V)$.

Definition 4. The kth exterior bundle over M (smooth manifold) is the vector bundle

$$\wedge^k(M) = \coprod_{x \in M} \wedge^k(T_x^\star M)$$

A section of the bundle $\wedge^k(M) \to M$ is called a differential k-form. The set of differential k-forms is denoted by $\Omega^k(M)$ and the set of differential forms $\bigoplus_k \Omega^k(M)$ is denoted by $\Omega(M)$, where $\Omega(M)$ has the structure of a module over the ring of smooth functions and of a graded algebra with wedge multiplication.

Theorem 5. Suppose ω is a differential 1-form on a smooth manifold M. X and Y are smooth tangent vector fields on M. Then

$$< X \wedge Y, d\omega > = X < Y, \omega > - Y < X, \omega > - < [X, Y], \omega >$$

Proof. Given that

$$< X \wedge Y, d\omega > = X < Y, \omega > - Y < X, \omega > - < [X, Y], \omega > \qquad (1)$$

Since both sides of equation (1) are linear with respect to ω , we may assume that ω is a monomial

$$\omega = g\, df \quad ; \text{where } f \text{ and } g \text{ are smooth functions on } M$$
$$\Rightarrow d\omega = dg \wedge df$$

L.H.S: $< X \wedge Y, d\omega >$

$$= < X \wedge Y, dg \wedge df >$$

$$= \begin{vmatrix} < X, dg > & < X, df > \\ < Y, dg > & < Y, df > \end{vmatrix}$$
$$= \begin{vmatrix} Xg & Xf \\ Yg & Yf \end{vmatrix}$$
$$= Xg.Yf - Xf.Yg$$

R.H.S: $X < Y, \omega > - Y < X, \omega > - < [X, Y], \omega >$

$$= X < Y, g\, df > - Y < X, g\, df > - < [X, Y], g\, df >$$

$$= X(g\, Yf) - Y(g\, Xf) - g[X, Y]f$$

$$= Xg.Yf + g\, XYf - Yg.Xf - g\, YXf - g\, XYf + g\, YXf$$

$$= Xg.Yf - Xf.Yg$$

Therefore L.H.S = R.H.S

This completes the proof of the theorem. □

Theorem 6. Suppose $f: M \to N$ is a smooth map from a smooth manifold M to a smooth manifold N. Then the induced map $f^*: A(N) \to A(M)$ commutes with the exterior derivative d, that is,

$$f^* o\ d = d\ o\ f^*: A(N) \to A(M) \qquad (2)$$

Proof. Since both f^* and d are linear, we need only consider the operation of both sides of (2) on a monomial β.

First suppose β is a smooth function on N i.e., $\beta \in A^0(N)$. Choose any smooth tangent vector field X on M. Then

$$\begin{aligned} < X, f^*(d\beta) > &= < f_* X, d\beta > \\ &= f_* X(\beta) = X(\beta\ o\ f) \\ &= < X, d(f^*\beta) >. \end{aligned}$$

Therefore $f^*(d\beta) = d(f^*\beta)$.

Next suppose $\beta = u\, dv$, where u, v are smooth functions on N.

Then

$$\begin{aligned} f^*(d\beta) &= f^*(du \wedge dv) \\ &= f^*du \wedge f^*dv = d(f^*u) \wedge d(f^*v) \\ &= d(f^*\beta). \end{aligned}$$

Now assume that (2) holds for exterior differential forms of degree $< r$. We need to show that it also holds for exterior differential r-forms. Suppose $\beta = \beta_1 \Lambda \beta_2$, where β_1 is a differential 1-form on N and β_2 is an exterior differential $(r-1)$ form on N. Then by the induction hypothesis we have

$$\begin{aligned} d\ o\ f^*(\beta_1 \wedge \beta_2) &= d(f^*\beta_1 \wedge f^*\beta_2) \\ &= d(f^*\beta_1) \wedge f^*\beta_2 - f^*\beta_1 \wedge d(f^*\beta_2) \\ &= f^*(d\beta_1 \wedge \beta_2) - f^*(\beta_1 \wedge d\beta_2) \\ &= f^* o\ d(\beta_1 \wedge \beta_2). \end{aligned}$$

This completes the proof of the theorem. □

4. Applications in differential geometry

The exterior algebra has notable applications in differential geometry, where it is used to define differential forms [4]. A differential form at a point of a differentiable manifold is an alternating multilinear form on the tangent space at the point. Equivalently, a differential form of degree k is a linear functional on the k-th exterior power of the tangent space. As a consequence, the wedge product of multilinear forms defines a natural wedge product for differential forms. Differential forms play a major role in diverse areas of differential geometry [6].

Let V be an n-dimensional real inner product space. We extend inner product from V to all of $\Lambda(V)$ by setting the inner product of elements which are homogeneous of different degrees equal to zero and by setting

$$< w_1 \wedge \ldots \wedge w_p, v_1 \wedge \ldots \wedge v_p > = det < w_i, v_j >$$

and then extending bilinearly to all of $\wedge^p(V)$.

If $e_1, \ldots, e_n$ is an orthonormal basis of V, then the corresponding basis $\{\{e_{i_1} \wedge \ldots \wedge e_{i_k}\}\ 1 \le i_1 < \cdots < i_k \le n$ of $\wedge(V)$ is an orthonormal basis for $\wedge(V)$.

Since $\wedge^n(V)$ is one dimensional, $\wedge^n(V) - \{0\}$ has two components. An orientation on V is a choice of a component of $\wedge^n(V) - \{0\}$. If V is an oriented inner product space, then there is a linear transformation

$$*: \Lambda(V) \to \Lambda(V)$$

called star, which is well defined by the requirement that for any orthonormal basis $e_1, \dots, e_n$ of V.

Theorem 7. Prove that on $\wedge^p(\mathrm{V})$, $\star\star = (-1)^{\mathrm{p(n-p)}}$. Also prove that for arbitrary $v, w \in \wedge^p(\mathrm{V})$, their inner product is given by $< v, w >=\star(w \wedge \star v) = \star(\star v \wedge \star w)$.

Proof. First we need to show that if $e_1, \dots, e_n$ is an orthonormal basis of Vm then $\{\{e_{i_1} \wedge \dots \wedge e_{i_k}\}\ 1 \le i_1 < \dots < i_k \le n\}$ is an orthonormal basis for $\wedge(V)$. To do this we just need to prove that

$$< e_{i_1} \wedge \dots \wedge e_{i_p}, e_{j_1} \wedge \dots \wedge e_{j_p} >= \delta_{i_1.j_1} \dots \delta_{i_p.j_p} = \begin{cases} 1 & if\ i_1 = j_1, \dots i_p = j_p \\ 0 & else \end{cases}$$

where $\delta_{i,j}$ is the Kronecker symbol.

It is to see that the matrix $(< e_{i_k}, e_{i_l} >)$ is the identity matrix, thus

$$< e_{i_1} \wedge \dots \wedge e_{i_p}, e_{i_1} \wedge \dots \wedge e_{i_p} > = \det(\text{Identity matrix of size } p \times p) = 1.$$

On the other hand, if $e_{i_1} \wedge \dots \wedge e_{i_p} \ne e_{j_1} \wedge \dots \wedge e_{j_p}$ that means there is a k such that $i_k \ne j_k$. This means, the kth row (column) of the matrix $(< e_{i_k}, e_{i_l} >)$ is zero.

Now we need to show that $\star\star = (-1)^{\mathrm{p(n-p)}}$ in $\wedge^p(\mathrm{V})$. We can assume without loss of generality that $\star\, 1 = \mathrm{e}_1 \wedge \dots \wedge e_n$. Note that the definition of $\star$ implies that

$$e_{i_1} \wedge \dots \wedge e_{i_p} \wedge \star\left(e_{i_1} \wedge \dots \wedge e_{i_p}\right) = \mathrm{e}_1 \wedge \dots \wedge e_n$$

$$e_{i_1} \wedge \dots \wedge e_{i_p} \wedge *(e_{i_1} \wedge \dots \wedge e_{i_p}) = \underbrace{*(e_{i_1} \wedge \dots \wedge e_{i_p})}_{(i)} \wedge \underbrace{**(e_{i_1} \wedge \dots \wedge e_{i_p})}_{(ii)}$$

then we have to do is to move (i), $n-p$ slots (because $(i)$is in $\wedge^{n-p}(\mathrm{V})$). So we get a $(-1)^{(\mathrm{n-p})}$ for each of the p basis elements that (ii) (because $(ii)$is in $\wedge^p(\mathrm{V})$). Since this is true for any oriented basis, we have that $\star\star = (-1)^{\mathrm{p(n-p)}}$.

Finally we need to show that $< v, w >=\star(w \wedge \star v) = \star(\star v \wedge \star w)$. Again we can simply work with monomials.

$$\star\left(e_{i_1} \wedge \dots \wedge e_{i_p} \wedge \star\left(e_{i_1} \wedge \dots \wedge e_{i_p}\right)\right) = \star(\mathrm{e}_1 \wedge \dots \wedge e_n) = 1 = \\ < e_{i_1} \wedge \dots \wedge e_{i_p}, e_{i_1} \wedge \dots \wedge e_{i_p} >$$

And on the other hand, if there is a k such that $i_k \ne j_k$, then $\star\left(e_{j_1} \wedge \dots \wedge e_{j_p}\right)$ will be of the form $\pm e_{ik} \wedge$ (something) which means that

$$\star\left(e_{i_1} \wedge \dots \wedge e_{i_p} \wedge \star\left(e_{i_1} \wedge \dots \wedge e_{i_p}\right)\right) = 0 =< e_{i_1} \wedge \dots \wedge e_{i_p}, e_{i_1} \wedge \dots \wedge e_{i_p} >$$

This complete the proof of the theorem □

Theorem 8. If ω is a p-form and η is a *(p+1)*-form then $(d\omega, \eta) = (\omega, \delta\eta)$.

Proof. We integrate over the closed manifold M the relation

$$d\omega \wedge * \eta + (-1)^p \omega \wedge d * \eta = d(\omega \wedge * \eta)$$

$$\int_M d\omega \wedge * \eta + (-1)^p \int_M \omega \wedge d * \eta = \int_M d(\omega \wedge * \eta) = \int_{\partial M} \omega \wedge * \eta = 0$$
$$(d\omega, \eta) = (-1)^{p-1} \int_M \omega \wedge d * \eta$$

Since $d * \eta$ is an $(n-p)$ −form we have

$$** (d * \eta) =* (* d * \eta) = (-1)^{p(n-p)} d * \eta$$

so that $$(-1)^{p-1} d * \eta = (-1)^{p-1}(-1)^{p(n-p)} * (* d * \eta)$$
$$= (-1)^{np+1} * (* d * \eta)$$

But η is a *(p+1)*-form, hence

$$\delta\eta = (-1)^{n(p+1)+n+1} * d * \eta = (-1)^{np+1} * (* d * \eta)$$

and so $$(-1)^{p-1} d * \eta =* \delta\eta$$
$$(d\omega, \eta) = (-1)^{p-1} \int_M \omega \wedge d * \eta$$

$$= \int_M \omega \wedge * (\delta\eta) = (\omega, \delta\eta)$$

This completes the proof of the theorem □

A form ω is called harmonic provided $\Delta\omega = 0$. It is clear that if the p −form ω satisfies the two equations $d\omega = 0,\ \delta\omega = 0$, then ω is harmonic. The converse is also true. Indeed, if ω is any p −form then

$$(\Delta\omega, \omega) = (d\delta\omega, \omega) + (d\delta\omega, \omega)$$
$$= (\delta\omega, \delta\omega) + (d\omega, d\omega)$$

Now if ω is harmonic then $\Delta\omega = 0$

$$(\delta\omega, \delta\omega) + (d\omega, d\omega) = 0$$

But each term is non-negative hence each vanishes, $(d\omega, d\omega) = 0, (\delta\omega, \delta\omega) = 0$ and this implies in turn that $d\omega = 0, \delta\omega = 0$.

Theorem 9. If ω is any p- form then there is a $(p-1)$-form α, a $(p+1)$-form β and α harmonic p- form γ such that $\omega = d\alpha + \delta\beta + \gamma$. The forms $d\alpha, \delta\beta, \gamma$ are unique.

Proof. We shall only settle the uniqueness part. Suppose we have

$$d\alpha + \delta\beta + \gamma = 0$$

We then have $d(d\alpha) = 0$ and also $d\gamma = 0$ since γ is harmonic. Hence

$$d\delta\beta = 0, (d\delta\beta, \beta) = 0, (\delta\beta, \delta\beta) = 0$$
$$\delta\beta = 0, d\alpha + \gamma, d\alpha = 0, \gamma = 0$$

By an almost identical argument one shows that in case ω is a closed p-form,$d\omega = 0$, then the term $\delta\beta$ in the Hodge decomposition of ω is absent $\omega = d\alpha + \gamma$

It follows from this that if z is any p-cycle then $\int_z \omega = \int_z \gamma$

That is, γ has the same period as does ω. The result of this is that is ω is any closed form then there exists a unique harmonic form γ with the same periods as those of ω.

This completes the proof of the theorem □

5. Generalized Stocks theorem

If M is an oriented k manifold with boundary ∂M and if α is a $(k-1)$-form defined on (an open set containing) M, then

$$\int_M d\alpha = \int_{\partial M} \alpha$$

Consider the 4-dimensional ball $B = \{(x, y, z, t) | x^2 + y^2 + z^2 + t^2 \leq R\}$ of radius R in $\mathbb{R}^4$ in two ways. This is a 4-manifold with boundary $S = \{(x, y, z, t) | x^2 + y^2 + z^2 + t^2 = R\}$.

V can be expressed as the integral $V = \iiint \int_B\ dx\, dy\, dz\, dt \int_B\ dx \wedge dy \wedge dz \wedge dt$

We use extended spherical co-ordinates $\sigma, \rho, \phi, \theta$, where σ measures the distance of (x, y, z, t) to the origin in $\mathbb{R}^4$ and ψ the angle to the t-axis. So that $t = \sigma \cos \psi$ and $\rho = \sigma \sin \psi$ is the distance from the projection (x, y, z) origin. Then letting θ, ϕ be the remaining spherical co-ordinates gives

$$x = \rho \sin \phi \cos \theta = \sigma \sin \psi \sin \phi \cos \theta$$
$$y = \rho \sin \phi \sin \theta = \sigma \sin \psi \sin \phi \sin \theta$$
$$z = \rho \cos \phi = \sigma sin\psi cos\phi$$

In these co-ordinates B is described as $\{0 \le \psi \le \pi, 0 \le \phi \le \pi, 0 \le \theta \le 2\pi, 0 \le \sigma \le R\}$. To simply computations, we note that form will get multiplied by the Jacobian when we change co-ordinates:

$$dx \wedge dy \wedge dz \wedge dt = \frac{\partial(x, y, z, t)}{\partial(\sigma, \rho, \phi, \theta)} d\sigma \wedge d\psi \wedge d\theta \wedge d\phi$$
$$= \sigma^3 sin^2\psi \sin \phi \, d\sigma \wedge d\psi \wedge d\theta \wedge d\phi$$

The volume is now easily computed $\int_0^R \sigma^3 \, d\sigma \int_0^\pi sin^2\psi d\psi \int_0^{2\pi} d\theta \int_0^\pi \sin \phi d\phi = \frac{1}{2}\pi^2 R^4$.

Alternatively we can use Stocks theorem $V = \int_B \, dx \wedge dy \wedge dz \wedge dt = -\int_S \, tdx \wedge dy \wedge dz$

The parameter x, y, z gives a left hand co-ordinate system on the upper hemisphere $U = S \cap \{t > 0\}$. It is left handed because t, x, y, z is left handed on $\mathbb{R}^4$. For similar reasons x, y, z gives a right handed system on the lower hemisphere L where $t < 0$. Therefore

$$V = -\int_U \, tdx \wedge dy \wedge dz - \int_L \, tdx \wedge dy \wedge dz$$
$$= \iiint_{x^2+y^2+z^2 \le R} \sqrt{R^2 - x^2 - y^2 - z^2} \, dxdydz$$
$$= 2\int_0^R \rho^2 \sqrt{R^2 - \rho^2} \, d\rho \, \int_0^\pi \sin \phi d\phi \, \int_0^{2\pi} d\theta$$
$$= 8\pi \int_0^{\frac{\pi}{2}} R^4 \, sin^2 \, \alpha cos^2 \, \alpha d\alpha$$
$$= \frac{1}{2}\pi^2 R^4$$

6. Laplace's equation

The Laplacian is a partial differential operator defined by

$$\Delta f = \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} + \frac{\partial^2 f}{\partial z^2}$$

This can be expressed using operators as $\Delta f = \nabla.(\nabla f)$. Suppose that F is a gravitational force this is known to be conservative so that $F = -\nabla P$. Substituting into $\nabla. F = -4\pi\rho$ yields the Poisson equation $\Delta P = 4\pi\rho$. In a Vacuum this reduces to Laplace's equation $\Delta P = 0$.

A solution to Laplace's equation is called a harmonic function. These are of fundamental importance both in pure and applied mathematics. If we write $r = \sqrt{x^2 + y^2 + z^2}$ then

$$P = -\frac{m}{r} + Constant$$

is the potential energy associated to particle of mass m.

We express Δ in terms of forms as $\Delta f dx \wedge dy \wedge dz = d \star df$ or $\Delta f = \star d \star df$

once we define $\star(gdx \wedge dy \wedge dz) = g$. This last formula also works in the plane provided we define

$$\star(fdx + gdy) = fdy - gdx$$
$$\star fdx \wedge dy = f$$

The $\star$ operator in n dimensions always takes p-forms to $(n-p)$- forms.

We have to find out the Laplace equation [11] in polar co-ordinates and use this to determine the radially symmetric harmonic functions on the plane. The key is the determination of the $\star$ −operator:

$$dr = \frac{\partial r}{\partial x}dx + \frac{\partial r}{\partial y}dy = \frac{x}{r}dx + \frac{y}{r}dy$$

Similarly $\quad d\theta = -\frac{y}{r^2}dx + \frac{x}{r^2}dy$

So that $\quad \star dr = \frac{x}{r}dy - \frac{y}{r}dx = rd\theta$

$$\star d\theta = -\frac{y}{r^2}dy - \frac{x}{r^2}dx = -\frac{1}{r}dr$$
$$\star dr \wedge d\theta = \star\frac{1}{r}dx \wedge dy = \frac{1}{r}$$

Thus $\quad \Delta f = \star d \star (\frac{\partial f}{\partial r}dr + \frac{\partial f}{\partial \theta}d\theta)$

$$= \star(r\frac{\partial f}{\partial r}d\theta - \frac{1}{r}\frac{\partial f}{\partial \theta}dr)$$
$$= \frac{1}{r}\frac{\partial f}{\partial r} + \frac{\partial^2 f}{\partial r^2} + \frac{1}{r^2}\frac{\partial^2 f}{\partial \theta^2}$$

If f is radially symmetric, then it depends only on r so we obtain

$$\frac{1}{r}\frac{df}{dr} + \frac{d^2y}{dr^2} = \frac{1}{r}\frac{d}{dr}\left(r\ \frac{df}{dr}\right) = 0$$

This differential equation can be solved using standard techniques to get

$$f(r) = C + D\log r \text{ for constants } C, D$$

By a similar calculation we find that $\quad f(r) = C + \frac{D}{r}$

are the only radially symmetric harmonic functions in $\mathbb{R}^3$.

7. Maxwell's equation in $\mathbb{R}^4$

In relativity theory [3] one needs to treat the electric $\boldsymbol{E} = E_1\boldsymbol{i} + E_2\boldsymbol{j} + E_3\boldsymbol{k}$ and magnetic field $\boldsymbol{B} = B_1\boldsymbol{i} + B_2\boldsymbol{j} + B_3\boldsymbol{k}$ as part of a single field on space time. In mathematical terms we can take space time to be $\mathbb{R}^4$ with the fourth coordinate as time t. The electromagnetic field can be represented by a 2-form

$$F = B_3dx \wedge dy + B_1dy \wedge dz + B_2dz \wedge dx + E_1dx \wedge dt + E_2dy \wedge dt + E_3dz \wedge dt$$

If we compute dF using the analogues of the rules we have

$$dF = \left(\frac{\partial B_3}{\partial x}dx + \frac{\partial B_3}{\partial y}dy + \frac{\partial B_3}{\partial z}dz + \frac{\partial B_3}{\partial t}dt\right) \wedge dx \wedge dy + \cdots$$
$$= \left(\frac{\partial B_1}{\partial x} + \frac{\partial B_2}{\partial y} + \frac{\partial B_3}{\partial z}\right)dx \wedge dy \wedge dz + \left(\frac{\partial E_2}{\partial x} - \frac{\partial E_1}{\partial y} + \frac{\partial B_3}{\partial t}\right)dx \wedge dy \wedge dt + \cdots$$

Two of Maxwell's equation's for electromagnetism

$$\nabla.\boldsymbol{B} = 0, \qquad \nabla \times E = -\frac{\partial \boldsymbol{B}}{\partial t}$$

can be expressed as $dF = 0$. The analogue of theorem 4.1 holds for $\mathbb{R}^n$, and shows that

$$F = d(A_1dx + A_2dy + A_3dz + A_4dt)$$

for some 1-form called the potential. In terms of vector analysis this amounts to the more complicated looking equations

$$\boldsymbol{B} = \nabla \times (A_1\boldsymbol{i} + A_2\boldsymbol{j} + A_3\boldsymbol{k}),\ \ \boldsymbol{E} = \nabla A_4 - \frac{\partial A_1}{\partial x}\boldsymbol{i} + \frac{\partial A_2}{\partial y}\boldsymbol{j} + \frac{\partial A_3}{\partial z}\boldsymbol{k}$$

There are two remaining Maxwell equations

$$\nabla.\boldsymbol{E} = 4\pi\rho, \quad \nabla \times \boldsymbol{B} = \frac{\partial \boldsymbol{E}}{\partial x} + 4\pi\boldsymbol{J}$$

where ρ is the electric charge density and $\boldsymbol{J}$ is the electric current. After applying the divergence theorem it implies that the electric flux through a closed surface equals (-4π) times the electric charge inside it. These last two Maxwell equations can also be replaced by the single equation $d \star F = 4\pi J$ of 3-forms. Here

$$\star F = E_3 dx \wedge dy + E_1 dy \wedge dz + E_2 dz \wedge dx - B_1 dx \wedge dt - B_2 dy \wedge dt - B_3 dz \wedge dt$$

and $$J = \rho dx \wedge dy \wedge dz - J_3 dx \wedge dy \wedge dt - J_1 dy \wedge dz \wedge ddt - J_2 dz \wedge dx \wedge dt$$

this implies that $dJ = \frac{1}{4\pi} d^2 \star F = 0$. Expanding this out yields

$$\frac{\partial \rho}{\partial t} dt \wedge dx \wedge dy \wedge dz - \frac{\partial J_3}{\partial z} dz \wedge dx \wedge dy \wedge dt + \cdots = -\left(\frac{\partial \rho}{\partial t} + \nabla.\boldsymbol{J}\right) dx \wedge dy \wedge dz \wedge dt = 0$$

To appreciate the meaning integrate $\frac{\partial \rho}{\partial t}$ over a solid region V with boundary S. Then this equals

$$-\iiint_V \nabla.\boldsymbol{J}\, dV = -\iint_S \boldsymbol{J}.\boldsymbol{n}\, ds$$

In other words the rate of change of the electric charge in V equals minus the flux of the current across the surface. This is the law of conservation of electric charge.

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