Annals of Pure and Applied Mathematics
Vol. 5, No. 2, 2014, 108-119
ISSN: 2279-087X (P), 2279-0888(online)
Published on 1 January 2014
www.researchmathsci.org

Annals of
Pure and Applied Mathematics

Certain Properties of Countably Compact Fuzzy Sets

M. A. M. Talukder[1] **and *D. M. Ali***[2]

[1] Department of Mathematics, Khulna University of Engineering & Technology, Khulna – 9203, Bangladesh.
Email: mottalib_kuet@yahoo.com

[2] Department of Mathematics, University of Rajshahi, Rajshahi – 6205, Bangladesh

Received 19 November 2013; accepted 6 December 2013

Abstract. In this paper, we study countably compact fuzzy sets using the definition of C. K. Wong [10] and obtain its several properties.

Keywords: Fuzzy topological spaces, countably compact fuzzy sets

AMS Mathematics Subject Classification (2010): 18B30

1. Introduction

The concept of fuzzy set and fuzzy set operations were first introduced by L. A. Zadeh [11] in1965. Several other mathematicians studied fuzzy sets in various areas in mathematics. Firstly, C. L. Chang [2] in 1968 developed the theory of fuzzy topological spaces. The purpose of this paper is to study countably compact fuzzy sets in the sense of C. K. Wong [10] and to obtain its several other properties.

2. Preliminaries

In this section, we recall some fundamental definitions which are needed in the sequel.. The following are essential in our study and can be found in the paper referred to.

Definition 2.1. [11] Let X be a non-empty set and I is the closed unit interval [0, 1]. A fuzzy set in X is a function $u : X \to I$ which assigns to every element $x \in X$. u(x) denotes a degree or the grade of membership of x. The set of all fuzzy sets in X is denoted by I^X. A member of I^X may also be called a fuzzy subset of X.

Definition 2.2. [8] A fuzzy set is empty iff its grade of membership is identically zero . It is denoted by 0 or ϕ.

Definition 2.3. [8] A fuzzy set is whole iff its grade of membership is identically one in X . It is denoted by 1 or X .

Definition 2.4. [2] Let u and v be two fuzzy sets in X. Then we define
(i) u = v iff u(x) = v(x) for all $x \in X$
(ii) $u \subseteq v$ iff $u(x) \leq v(x)$ for all $x \in X$
(iii) $\lambda = u \cup v$ iff $\lambda(x) = (u \cup v)(x) = \max[u(x), v(x)]$ for all $x \in X$

(iv) $\mu = u \cap v$ iff μ(x) = (u$\cap$v) (x) = min [u(x) , v(x)] for all x $\in$ X

(v) $\gamma = u^c$ iff γ(x) = 1 – u(x) for all x $\in$ X.

Definition 2.5. [2] In general , if { u_i : $i \in$ J } is family of fuzzy sets in X , then union $\cup u_i$ and intersection $\cap u_i$ are defined by

$\cup u_i$ (x) = sup { u_i (x) : $i \in$ J and x $\in$ X }

$\cap u_i$ (x) = inf { u_i (x) : $i \in$ J and x $\in$ X } , where J is an index set.

Definition 2.6. [2] Let f : X $\rightarrow$ Y be a mapping and u be a fuzzy set in X. Then the image of u , written $f(u)$, is a fuzzy set in Y whose membership function is given by

$$\text{f(u) (y)} = \begin{cases} \sup\{ u(x) : x \in f^{-1}(y)\} & if \;\; f^{-1}(y) \neq \phi \\ 0 & if \;\; f^{-1}(y) = \phi \end{cases} .$$

Definition 2.7. [2] Let f : X $\rightarrow$ Y be a mapping and v be a fuzzy set in Y. Then the inverse of v, written f^{-1}(v) , is a fuzzy set in X whose membership function is given by f^{-1}(v) (x) = v (f (x)).

Distributive laws 2.8. [11] Distributive laws remain valid for fuzzy sets in X i.e. if u , v and w are fuzzy sets in X , then

(i) u $\cup$ (v $\cap$ w) = (u $\cup$ v) $\cap$ (u $\cup$ w)

(ii) u $\cap$ (v $\cup$ w) = (u $\cap$ v) $\cup$ (u $\cap$ w) .

Definition 2.9. [1] Let λ be a fuzzy set in X , then the set { x $\in$ X : λ(x) > 0 } is called the support of λ and is denoted by λ_0 or suppλ.

Definition 2.10. [2] Let X be a non-empty set and t $\subseteq I^X$ i.e. t is a collection of fuzzy set in X. Then t is called a fuzzy topology on X if

(i) 0 , 1 $\in$ t

(ii) $u_i \in$ t for each $i \in$ J , then $\bigcup_i u_i \in$ t

(iii) u , v $\in$ t , then u $\cap$ v $\in$ t.

The pair (X , t) is called a fuzzy topological space and in short, fts. Every member of t is called a t-open fuzzy set. A fuzzy set is t-closed iff its complements is t-open. In the sequel, when no confusion is likely to arise, we shall call a t-open (t-closed) fuzzy set simply an open (closed) fuzzy set.

Definition 2.11. [2] Let (X , t) and (Y , s) be two fuzzy topological spaces. A mapping f : $(X , t) \rightarrow (Y , s)$ is called an fuzzy continuous iff the inverse of each s-open fuzzy set is t-open.

Definition 2.12. [8] Let (X, t) be an fts and A $\subseteq$ X. Then the collection t_A = { u|A : u $\in$ t } = { u $\cap$ A : u $\in$ t } is fuzzy topology on A, called the subspace fuzzy topology on A and the pair (A, t_A) is referred to as a fuzzy subspace of (X, t).

Definition 2.13. [3] Let (A, t_A) and (B, s_B) be fuzzy subspaces of fuzzy topological spaces (X, t) and (Y, s) respectively and f is a mapping from (X, t) to (Y, s), then we say that f is a mapping from (A, t_A) to (B, s_B) if f(A) $\subseteq$ B.

Definition 2.14. [3] Let (A, t_A) and (B, s_B) be fuzzy subspaces of fts's (X, t) and (Y, s) respectively. Then a mapping f : $(A, t_A) \to (B, s_B)$ is relatively fuzzy continuous iff for each v $\in$ s_B, the intersection f^{-1}(v) $\cap$ A $\in$ t_A.

Definition 2.15. [1] Let (X, T) be a topological space. A function f : X $\to$ **R** (with usual topology) is called lower semi-continuous (l . s . c.) if for each a $\in$ **R** , the set f^{-1}(a , ∞) $\in$ T. For a topology T on a set X , let ω(T) be the set of all l . s . c. functions from (X, T) to I (with usual topology); thus ω(T) = { u $\in$ I^X : u^{-1}(a , 1] $\in$ T , a $\in$ I_1 } . It can be shown that ω(T) is a fuzzy topology on X.

Definition 2.16. [9] An fts (X, t) is said to be fuzzy – T_1 space iff for every x , y $\in$ X , x $\neq$ y, there exist u , v $\in$ t such that u(x) = 1, u(y) = 0 and v(x) = 0, v(y) = 1.

Definition 2.17. [5] An fts (X, t) is said to be fuzzy Hausdorff iff for all x , y $\in$ X , x $\neq$ y, there exist u , v $\in$ t such that u(x) = 1 , v(y) = 1 and u $\cap$ v = 0 .

Definition 2.18. [7] An fts (X, t) is said to be fuzzy Hausdorff iff for all x , y $\in$ X , x $\neq$ y , there exist u , v $\in$ t such that u(x) = 1 , v(y) = 1 and u $\subseteq$ 1 – v.

Definition 2.19. [7] An fts (X, t) is said to be fuzzy regular iff for each x $\in$ X and u $\in$ t^c with u(x) = 0 , there exist v , w $\in$ t such that v(x) = 1 , u $\subseteq$ w and v $\subseteq$ 1 – w .

Definition 2.20. [2] A family F of fuzzy sets is a cover of a fuzzy set λ iff $\lambda \subseteq$ { u_i : $u_i \in$ F }. It is an open cover iff each member of F is an open fuzzy set. A subcover of F is a subfamily of F which is also a cover.

Definition 2.21. [9] A fuzzy set λ in X is said to be countably fuzzy compact iff every countable open cover of λ has a finite subcover.

3. Characterizations of countably compact fuzzy sets

In this section, we investigate some tangible properties of countably compact fuzzy sets.

Theorem 3.1. Let λ be a fuzzy set in an fts (X, t) and A $\subseteq$ X . Then the following are equivalent:

(i) λ is countably compact fuzzy set with respect to t .

(ii) λ is countably compact fuzzy set with respect to the subspace fuzzy topology t_A on A .

Proof : Suppose λ is countably compact fuzzy set with respect to t . Let M = { u_k : k $\in$ N } be a countable open cover of λ with respect to t_A. Then by definition of subspace fuzzy topology, there exists $v_k \in$ t such that u_k = A $\cap$ $v_k \subseteq v_k$ and hence we see that $\lambda \subseteq \bigcup_{k\in N} u_k \subseteq \bigcup_{k\in N} v_k$ and therefore { v_k : k $\in$ N } is a countable open cover of λ with respect to t. Since λ is countably compact fuzzy set with respect to t, so λ has a finite subcover, say $v_{k_r} \in$ { v_k } (r = 1, 2, …… , n) such that $\lambda \subseteq \bigcup_{r=1}^{n} v_{k_r}$. But $\lambda \subseteq$ A $\cap$ ($\bigcup_{r=1}^{n} v_{k_r}$) $\subseteq$ $\bigcup_{r=1}^{n}$(A $\cap$ v_{k_r}) $\subseteq$ $\bigcup_{r=1}^{n} u_{k_r}$. Therefore { u_k : $k \in$ N } contains a finite subcover { u_{k_r} } (r = 1, 2, …… , n). Hence λ is countably compact fuzzy set with respect to t_A.

Conversely, suppose λ is countably compact fuzzy set with respect to t_A. Let { v_k : $k \in$ N } be a countable open cover of λ with respect to t. Put u_k = A $\cap$ v_k , then $\lambda \subseteq \bigcup_{k\in N} v_k$ implies $\lambda \subseteq$ A $\cap$ ($\bigcup_{k\in N} v_k$) $\subseteq$ $\bigcup_{k\in N}$(A $\cap$ v_k) $\subseteq$ $\bigcup_{k\in N} u_k$. Since $u_k \in t_A$, so { u_k : $k \in$ N } is a countable open cover of λ with respect to t_A. As λ is countably compact fuzzy set with respect to t_A , then { u_k : $k \in$ N } has a finite subcover , say { u_{k_r} } (r = 1, 2, …… , n). Accordingly, $\lambda \subseteq u_{k_1} \cup u_{k_2} \cup$ ...... $\cup\, u_{k_n}$ = (A $\cap$ v_{k_1}) $\cup$ (A $\cap$ v_{k_2}) $\cup$ …… (A $\cap$ v_{k_n}) $\subseteq$ A $\cap$ ($v_{k_1} \cup v_{k_2} \cup$ ….. $\cup\, v_{k_n}$) $\subseteq$ $v_{k_1} \cup v_{k_2} \cup$ ….. $\cup\, v_{k_n}$. Therefore { v_k : $k \in$ N } contains a finite subcover { v_{k_r} } (r = 1, 2, …… , n) and hence λ is countably compact fuzzy set with respect to t .

Theorem 3.2. Let (X, t) and (Y, s) be two fuzzy topological spaces and f : $(X, t) \rightarrow (Y, s)$ be continuous and onto. If λ is countably compact fuzzy set in (X, t), then f(λ) is also countably compact fuzzy set in (Y, s).

Proof: Let { u_k : k $\in$ N } be a countable open cover of f(λ) in (Y, s). Since f is fuzzy continuous , then f^{-1}(u_k) $\in$ t and hence { f^{-1}(u_k) : k $\in$ N } is a countable open cover of λ. As λ is countably compact fuzzy set in (X, t), then λ has a finite subcover, say $\{\ f^{-1}(\ u_{k_r}\ ) : r\ = 1, 2, \ldots\ldots , n\ \}$ such that $\lambda \subseteq \bigcup_{r=1}^{n} f^{-1}(\ u_{k_r}\ )$. Again , let u be any fuzzy set in Y. Since f is onto , then for any y $\in$ Y, we have

f(f^{-1}(u)) (y) = sup { f^{-1}(u) (z) : z $\in$ f^{-1}(y) , f^{-1}(y) $\neq \phi$ } = sup {u(f(z)) : f(z) = y } = sup {u(y) }= u(y) i.e. f(f^{-1}(u)) = u . This is true for any fuzzy set in Y. Hence f(λ) $\subseteq$ f $(\bigcup_{r=1}^{n} f^{-1}(\ u_{k_r}\ )) = \bigcup_{r=1}^{n} f\ (\ f^{-1}(\ u_{k_r}\ )\ ) = \bigcup_{r=1}^{n} u_{k_r}$. Therefore f(λ) is countably compact fuzzy set in $(Y\ , s)$.

Theorem 3.3. Let $(X\ , t)$ and $(Y\ , s)$ be two fuzzy topological spaces and f : $(X\ , t)$ $\rightarrow$ $(Y\ , s)$ be bijective. If λ is countably compact fuzzy set in $(Y\ , s)$, then $f^{-1}(\ \lambda\ )$ is countably compact fuzzy set in $(X\ , t)$.

Proof : Let $\{\ u_k\ : k \in$ N } be a countable open cover of $f^{-1}(\ \lambda\ )$ in $(X\ , t)$. Then { f(u_k) : $k \in$ N } is a countable open cover of λ in $(Y\ , s)$. Since λ is countably compact fuzzy set in $(Y\ , s)$, then λ has a finite subcover i.e. there exist f(u_{k_r}) $\in$ { f(u_k) } (r = 1, 2, …… , n) such that $\lambda \subseteq \bigcup_{r=1}^{n} f\ (\ u_{k_r}\ )$. Again, let u be any fuzzy set in X. Since f is bijective, then for any x $\in$ X, we have

f^{-1}(f(u)) (x) = f(u) (f(x)) = u (f^{-1}(f(x))) = u(x). Thus f^{-1}(f(u)) = u and this is true for any fuzzy set in X. Hence $f^{-1}(\ \lambda\ ) \subseteq f^{-1}(\bigcup_{r=1}^{n} f\ (\ u_{k_r}\ )\ ) \subseteq \bigcup_{r=1}^{n} f^{-1}(\ f(u_{k_r})\ )$ $\subseteq \bigcup_{r=1}^{n} u_{k_r}$. Therefore $f^{-1}(\ \lambda\ )$ is countably compact fuzzy set in $(X\ , t)$.

Theorem 3.4. Let $(X\ , t)$ be an fts and $(A\ , t_A)$ be subspace of $(X\ , t)$ and f : $(X\ , t) \rightarrow (A\ , t_A)$ be continuous and onto. If λ is countably compact fuzzy set in $(X\ , t)$, then f (λ) is countably compact fuzzy set in $(A\ , t_A)$.

Proof : Let $\{\ u_k\ : k \in$ N } be a countable open cover of f (λ) in $(A\ , t_A)$ i.e. f (λ) $\subseteq \bigcup_{k \in N} u_k$. Since f is continuous, then $f^{-1}(u_k) \in$ t and consequently $\{\ f^{-1}(u_k) : k \in$ N } is a countable open cover of λ in $(X\ , t)$. As λ is countably compact fuzzy set in $(X\ , t)$, then λ has a finite subcover i.e. there exist $f^{-1}(u_{k_r}) \in \{\ f^{-1}(u_k)\ \}$ (r = 1, 2, …… , n) such that $\lambda \subseteq \bigcup_{r=1}^{n} f^{-1}(u_{k_r})$. Again , let u be any fuzzy set in A. Since f is onto , then for any y ∈ A, we have f (f^{-1}(u)) (y) = sup { f^{-1}(u) (z) : z ∈ f^{-1}(y) , f^{-1}(y) $\neq \phi$ } = sup {u (f(z)) : f(z) = y } = sup { u(y) } = u(y) i.e. f (f^{-1}(u)) = u . This is true for any fuzzy set in A. Hence f(λ) $\subseteq$ f $(\bigcup_{r=1}^{n} f^{-1}(u_{k_r})) = \bigcup_{r=1}^{n} f(f^{-1}(u_{k_r}))$ $= \bigcup_{r=1}^{n} u_{k_r}$. Therefore f(λ) is countably compact fuzzy set in (A, t_A).

Theorem 3.5. Let (A, t_A) and (B, s_B) be fuzzy subspaces of fuzzy topological spaces (X, t) and (Y, s) respectively. Let λ be a countably compact fuzzy set in (A, t_A) and f : $(A, t_A) \to (B, s_B)$ be relatively fuzzy continuous and onto. Then f (λ) is also countably compact fuzzy set in (B, s_B).

Proof : Let { v_k : $k \in$ N } be a countable open cover of f (λ) in (B, s_B) i.e. f (λ)$\subseteq \bigcup_{k\in N} v_k$. Since $v_k \in s_B$, then there exists $u_k \in$ s such that $v_k = u_k \cap$ B. Hence f (λ) $\subseteq \bigcup_{k\in N}(u_k \cap$ B). As f is relatively fuzzy continuous, then $f^{-1}(v_k) \cap$ A ∈ t_A and hence { $f^{-1}(v_k) \cap$ A : $k \in$ N } is a countable open cover of λ in (A, t_A) i.e. { $f^{-1}(u_k \cap$ B) ∩ A : $k \in$ N } = { $f^{-1}(u_k) \cap f^{-1}$(B) ∩ A : $k \in$ N } = { $f^{-1}(u_k) \cap$ A : $k \in$ N } is a countable open cover of λ in (A, t_A). As λ is countably compact fuzzy set in (A, t_A), then λ has a finite subcover i.e. there exist $f^{-1}(u_{k_r})$ ∩ A ∈ { $f^{-1}(u_k)$ } (r = 1, 2, …… , n) such that $\lambda \subseteq \bigcup_{r=1}^{n}(f^{-1}(u_{k_r}) \cap$ A). Again, let u be any fuzzy set in B. Since f is onto , then for any y ∈ B, we have f (f^{-1}(u)) (y) = sup { f^{-1}(u) (z) : z ∈ f^{-1}(y) , f^{-1}(y) $\neq \phi$ } = sup { u (f(z)) : f(z) = y } = sup { u(y) } = u(y) i.e. f (f^{-1}(u) = u and this is true for any fuzzy set in B. Hence f (λ)$\subseteq$ f ($\bigcup_{r=1}^{n}(f^{-1}(u_{k_r}) \cap$ A)) = $\bigcup_{r=1}^{n} f(f^{-1}(u_{k_r}) \cap$ A)) = $\bigcup_{r=1}^{n}(u_{k_r} \cap$ f (A)) = $\bigcup_{r=1}^{n}(u_{k_r} \cap$ B) = $\bigcup_{r=1}^{n} v_{k_r}$. Therefore f (λ) is countably compact fuzzy set in (B, s_B).

Theorem 3.6. Let λ be a countably compact fuzzy set in a fuzzy – T_1 space (X , t) with $\lambda_0 \subset$ X (proper subset). Let x $\notin \lambda_0$ (λ(x) = 0) , then there exist u , v $\in$ t such that u(x) = 1 , $\lambda_0 \subseteq v^{-1}(0, 1]$.

Proof : Let y $\in \lambda_0$. Then clearly x $\neq$ y. As (X , t) is fuzzy – T_1 space, then there exist u_y , $v_y \in$ t such that u_y(x) = 1 , u_y(y) = 0 and v_y(x) = 0 , v_y(y) = 1. Thus we see that $\lambda \subseteq \bigcup\{ v_y$: y$\in \lambda_0$ } i.e. { v_y : y $\in \lambda_0$ } is a countable open cover of λ in (X , t). But λ is countably compact fuzzy set, so λ has a finite subcover i.e. there exist $v_{y_k} \in$ { v_y } (k = 1, 2, …… , n) such that $\lambda \subseteq \bigcup_{k=1}^{n} v_{y_k}$. Now, let v = $v_{y_1} \cup v_{y_2} \cup ...... \cup v_{y_n}$ and u = $u_{y_1} \cap u_{y_2} \cap ...... \cap u_{y_n}$. Hence we see that v and u are open fuzzy sets, as they are the union and finite intersection of open fuzzy sets respectively i.e. v, u $\in$ t. Furthermore, $\lambda \subseteq v^{-1}(0, 1]$ and u(x) = 1, as u_{y_k} (x) = 1 for each k.

Theorem 3.7. Let λ and μ be disjoint countably compact fuzzy sets in a fuzzy – T_1 space (X , t) with λ_0 , $\mu_0 \subset$ X (proper subsets). Then there exist u , v $\in$ t such that $\lambda_0 \subseteq u^{-1}(0, 1]$ and $\mu_0 \subseteq v^{-1}(0, 1]$.

Proof: Let y $\in \lambda_0$. Then y $\notin \mu_0$, as λ and μ are disjoint. Since μ is countably compact fuzzy set in (X , t), then by previous theorem 3.6, there exist u_y , $v_y \in$ t such that u_y(y) = 1 and $\mu_0 \subseteq v_y^{-1}(0, 1]$. As u_y(y) = 1, then { u_y : y $\in \lambda_0$ } is a countable open cover of λ. But λ is countably compact fuzzy set in (X , t), then λ has a finite subcover i.e. there exist $u_{y_k} \in$ { u_y } (k = 1, 2, …… , n) such that $\lambda \subseteq \bigcup_{k=1}^{n} u_{y_k}$. Furthermore, $\mu \subseteq \bigcap_{k=1}^{n} v_{y_k}$, as $\mu \subseteq v_{y_k}$ for each k. Now, let u = $u_{y_1} \cup u_{y_2} \cup ...... \cup u_{y_n}$ and v = $v_{y_1} \cap v_{y_2} \cap ...... \cap v_{y_n}$. Hence we see that $\lambda_0 \subseteq u^{-1}(0, 1]$ and $\mu_0 \subseteq v^{-1}(0, 1]$. Thus u and v are open fuzzy sets, as they are the union and finite intersection of open fuzzy sets respectively i.e. u, v $\in$ t.

Remark : If λ(x) $\neq$ 0 for all x $\in$ X i.e. λ_0 = X, then the above two theorems 3.6 and 3.7 are not at all true.

Theorem 3.8. Let λ be a countably compact fuzzy set in a fuzzy Hausdorff space (X, t) (in the sense of Definition 2.17) with $\lambda_0 \subset$ X (proper subset). Let x$\notin \lambda_0$ (λ(x) = 0) , then there exist u , v $\in$ t such that u(x) = 1 , $\lambda_0 \subseteq v^{-1}(0, 1]$ and u $\cap$ v = 0 .

Proof : Let y $\in \lambda_0$. Then clearly x $\neq$ y . Since (X, t) is fuzzy Hausdorff , then there exist u_y , $v_y \in$ t such that u_y(x) = 1 , v_y(y) = 1 and $u_y \cap v_y = 0$. Thus we see that $\lambda \subseteq \cup\{ v_y : y \in \lambda_0 \}$ i.e. $\{ v_y : y \in \lambda_0 \}$ is a countable open cover of λ. As λ is countably compact fuzzy set in (X, t) , then λ has a finite subcover i.e. there exist $v_{y_r} \in \{ v_y \}$ (r = 1, 2, …… , n) such that $\lambda \subseteq \bigcup_{r=1}^{n} v_{y_r}$. Now, let v = $v_{y_1} \cup v_{y_2} \cup$ ….. $\cup\, v_{y_n}$ and u = $u_{y_1} \cap u_{y_2} \cap$ …… $\cap\, u_{y_n}$. Hence v and u are open fuzzy sets, as they are the union and finite intersection of open fuzzy sets respectively i.e. v , u $\in$ t . Furthermore, $\lambda_0 \subseteq v^{-1}(0, 1]$ and u(x) = 1 , since u_{y_r} (x) = 1 for each r .
Finally, we have to show that u $\cap$ v = 0. First, we observe that $u_{y_r} \cap v_{y_r}$ = 0 implies that u $\cap\, v_{y_r}$ = 0, by distributive law , we see that u $\cap$ v = u $\cap$($v_{y_1} \cup v_{y_2} \cup$ ….. $\cup\, v_{y_n}$) = 0 .

Theorem 3.9. Let λ and μ be disjoint countably compact fuzzy sets in a fuzzy Hausdorff space (X, t) (in the sense of Definition 2.17) with λ_0 , $\mu_0 \subset$ X (proper subsets) . Then there exist u , v $\in$ t such that $\lambda_0 \subseteq u^{-1}(0, 1]$, $\mu_0 \subseteq v^{-1}(0, 1]$ and u $\cap$ v = 0 .

Proof: Let y $\in \lambda_0$. Then y $\notin \mu_0$, as λ and μ are disjoint . As μ is countably compact fuzzy set in (X, t) , then by theorem 3.8, there exist u_y , $v_y \in$ t such that u_y(y) = 1 , $\mu_0 \subseteq v_y^{-1}(0, 1]$ and $u_y \cap v_y = 0$. Since u_y(y) = 1 , then $\{ u_y : y \in \lambda_0 \}$ is a countable open cover of λ . As λ is countably compact fuzzy set in (X, t) , then λ has a finite subcover i.e. there exist $u_{y_r} \in \{ u_y \}$ (r = 1, 2, …… , n) such that $\lambda \subseteq \bigcup_{r=1}^{n} u_{y_r}$. Furthermore , $\mu \subseteq \bigcap_{r=1}^{n} v_{y_r}$, as $\mu \subseteq v_{y_r}$ for each r . Now , let u = $u_{y_1} \cup u_{y_2} \cup$ …… $\cup\, u_{y_n}$ and v = $v_{y_1} \cap v_{y_2} \cap$ ….. $\cap\, v_{y_n}$. Thus we see that $\lambda_0 \subseteq u^{-1}(0, 1]$ and $\mu_0 \subseteq v^{-1}(0, 1]$. Hence u and v are open fuzzy sets , as they are the union and finite intersection of open fuzzy sets respectively i.e. u , v $\in$ t .

Finally, we have to show that u $\cap$ v = 0. First, we observe that $u_{y_r} \cap v_{y_r}$ = 0 implies that $u_{y_r} \cap$ v = 0, by distributive law, we see that u $\cap$ v = ($u_{y_1} \cup u_{y_2} \cup \ldots\ldots \cup u_{y_n}$) $\cap$ v = 0 .

Remark : If λ(x) $\neq$ 0 for all x $\in$ X i.e. λ_0 = X, then the above two theorems 3.8 and 3.9 are not at all true.

The following example will show that the countably compact fuzzy set in a fuzzy Hausdorff space (X , t) (in the sense of Definition 2.17) need not be closed.

Example 3.10. Let X = { a, b } and I = [0 , 1]. Let u_1, u_2 $\in$ I^X defined by u_1(a) = 1 , u_1(b) = 0 , and u_2(a) = 0 , u_2(b) = 1. Take t = { 0, 1, u_1 , u_2 }, then we see that (X , t) is fuzzy Hausdorff space. Let λ $\in$ I^X defined by λ(a) = 0.4, λ(b) = 0.6. Hence by definition of countably compact fuzzy set, we observe that λ is countably compact fuzzy set in (X , t). But λ is not closed, as its complement λ^c is not open in (X , t).

Theorem 3.11. Let λ be a countably compact fuzzy set in a fuzzy Hausdorff space (X , t) (in the sense of Definition 2.18) with λ_0 $\subset$ X (proper subset) . Let x$\notin$ λ_0 (λ(x) = 0) , then there exist u , v $\in$ t such that u(x) = 1 , $\lambda_0 \subseteq v^{-1}$(0 , 1] and u $\subseteq$ 1 – v.

Proof: Let y $\in$ λ_0. Then clearly x $\neq$ y . As (X , t) is fuzzy Hausdorff, then there exist u_y , v_y $\in$ t such that u_y(x) = 1, v_y(y) = 1 and $u_y \subseteq 1 - v_y$. Thus we see that $\lambda \subseteq \cup${ v_y : y $\in$ λ_0 } i.e. { v_y : y $\in$ λ_0 } is a countable open cover of λ. Since λ is countably compact fuzzy set in (X , t), then λ has a finite suvcover, i.e. there exist $v_{y_k} \in \{ v_y \}$ (k = 1, 2, …… , n) such that $\lambda \subseteq \bigcup_{k=1}^{n} v_{y_k}$. Now, let v = $v_{y_1} \cup v_{y_2} \cup \ldots\ldots \cup v_{y_n}$ and u = $u_{y_1} \cap u_{y_2} \cap \ldots\ldots \cap u_{y_n}$. Thus v and u are open fuzzy sets , as they are the union and finite intersection of open fuzzy sets respectively i.e. v , u $\in$ t. Furthermore , $\lambda_0 \subseteq v^{-1}$(0 , 1] and u(x) = 1 , as u_{y_k} (x) = 1 for each k .

Lastly, we have to show that u $\subseteq$ 1 – v . First, we observe that $u_y \subseteq 1 - v_y$ implies that u $\subseteq 1 - v_y$. Since u_{y_k} (x) $\leq 1 - v_{y_k}$ (x) for all x $\in$ X and for each k , then u $\subseteq$ 1 – v . If not , then there exist x $\in$ X , such that u_y(x) $\leq 1 - v_y$(x) . We have u_y(x)

$\leq u_{y_k}$ (x) for each k. Then for some k , u_{y_k} (x) $\leq 1 - v_{y_k}$ (x) . But this is a contradiction , as u_{y_k} (x) $\leq 1 - v_{y_k}$ (x) for each k . Hence u $\subseteq$ 1 – v .

Theorem 3.12. Let λ and μ be disjoint countably compact fuzzy sets in a fuzzy Hausdorff space (X , t) (in the sense of Definition 2.18) with λ_0 , $\mu_0 \subset$ X (proper subsets) . Then there exist u , v $\in$ t such that $\lambda_0 \subseteq u^{-1}(0 , 1]$, $\mu_0 \subseteq v^{-1}(0 , 1]$ and u $\subseteq$ 1 – v.

Proof: Let y $\in \lambda_0$. Then y $\notin \mu_0$, as λ and μ are disjoint. Since μ is countably compact fuzzy set in (X , t), then by previous theorem (3.11) , there exist u_y , $v_y \in$ t such that u_y(y) = 1, $\mu_0 \subseteq v_y^{-1}(0 , 1]$ and $u_y \subseteq 1 - v_y$. As u_y(y) = 1, then { u_y : y $\in \lambda_0$ } is a countable open cover of λ. Since λ is countably compact fuzzy set in (X , t), then λ has a finite suvcover i.e. there exist $u_{y_k} \in \{u_y\}$ (k = 1, 2, …… , n) such that $\lambda \subseteq \bigcup_{k=1}^{n} u_{y_k}$. Furthermore , $\mu \subseteq \bigcap_{r=1}^{n} v_{y_k}$, as $\mu \subseteq v_{y_k}$ for each k . Now, let u = $u_{y_1} \cup u_{y_2} \cup \ldots\ldots \cup u_{y_n}$ and v = $v_{y_1} \cap v_{y_2} \cap \ldots.. \cap v_{y_n}$. Hence $\lambda_0 \subseteq u^{-1}(0 , 1]$ and $\mu_0 \subseteq v^{-1}(0 , 1]$. Thus u and v are open fuzzy sets , as they are the union and finite intersection of open fuzzy sets respectively i.e. u , v $\in$ t .
Finally, we have to show hat u $\subseteq$ 1 – v . First we observe that $u_{y_k} \subseteq 1 - v_{y_k}$ for each k implies that $u_{y_k} \subseteq$ 1– v for each k and it is clear that u $\subseteq$ 1 – v.

Remark : If λ(x) $\neq$ 0 for all x $\in$ X i.e. λ_0 = X, then the above two theorems 3.11 and 3.12 are not at all true.

The following example will show that the countably compact fuzzy set in a fuzzy Hausdorff space (X , t) (in the sense of Definition 2.18) need not be closed.
Example 3.10 will surve the purpose.

Theorem 3.13. Let λ be a countably compact fuzzy set in a fuzzy regular space (X , t) with $\lambda_0 \subset$ X (proper subset). If for each x $\in \lambda_0$, there exist u $\in t^c$ with u(x) = 0, we have v , w $\in$ t such that v(x) = 1, u $\subseteq$ w, $\lambda_0 \subseteq v^{-1}(0 , 1]$ and v $\subseteq$ 1 – w.

Proof: Let (X , t) be a fuzzy regular space and λ be a countbly compact fuzzy set in X. Then for each x $\in \lambda_0$, there exists u $\in t^c$ with u(x) = 0. As (X , t) is fuzzy regular, we have v_x , $w_x \in$ t such that v_x(x) = 1, $u_x \subseteq w_x$ and $v_x \subseteq 1 - w_x$. Thus we see that

$\lambda \subseteq \bigcup\{ v_x : x \in \lambda_0 \}$ i.e. $\{ v_x : x \in \lambda_0 \}$ is a countable open cover of λ. Since λ is countably compact fuzzy set in X, then λ has a finite subcover, i.e. there exist $v_{x_r} \in \{ v_x \}$ (r = 1, 2, …… , n) such that $\lambda \subseteq \bigcup_{r=1}^{n} v_{x_r}$. Now, let v = $v_{x_1} \cup v_{x_2} \cup \ldots\ldots \cup v_{x_n}$ and w = $w_{x_1} \cap w_{x_2} \cap \ldots\ldots \cap w_{x_n}$. Hence we see that v and w are open fuzzy sets, as they are the union and finite intersection of open fuzzy sets respectively i.e. v, w $\in$ t. Furthermore, $\lambda_0 \subseteq v^{-1}(0, 1]$, v(x) = 1 and u $\subseteq$ w, as u $\subseteq w_{x_r}$ for each r.

Lastly, we have to show that v $\subseteq$ 1 – w. First, we observe that $v_{x_r} \subseteq 1 - w_{x_r}$ for each r implies that $v_{x_r} \subseteq$ 1 – w for each r and hence it is clear that v $\subseteq$ 1 – w.

Theorem 3.14. Let (X, T) be a topological space and $(X, \omega(T))$ be an fts. Let λ be a countably compact fuzzy set in $(X, \omega(T))$. Then $\lambda^{-1}(0, 1]$ is countably compact in (X, T). The converse is not true.

Proof : Suppose λ be countably compact fuzzy set in $(X, \omega(T))$. Let $\{ V_k : k \in$ N } be a countable open cover of $\lambda^{-1}(0, 1]$ in (X, T) i.e. $\lambda^{-1}(0, 1] \subseteq \bigcup_{k \in N} V_k$. As 1_{V_k} is l.s.c, then $1_{V_k} \in \omega$(T) and $\{ 1_{V_k} : 1_{V_k} \in \omega(T) \}$ is a countable open cover of λ in $(X, \omega(T))$. Since λ is countably compact fuzzy set in $(X, \omega(T))$, then λ has a finite subcover i.e. there exist $1_{V_{k_r}} \in \{ 1_{V_k} \}$ (r = 1, 2, …… , n) such that $\lambda \subseteq \bigcup_{r=1}^{n} 1_{V_{k_r}}$. Now, we can write $\lambda^{-1}(0, 1] \subseteq \bigcup_{r=1}^{n} 1_{V_{k_r}}$ and therefore $\lambda^{-1}(0, 1]$ is countably compact in (X, T).

Conversely, let X = { a, b, c } and T = { {a}, {b}, {a, b}, ϕ , X }, then (X, T) is a topological space. Let $u_1, u_2, u_3 \in I^X$ with u_1(a) = 0.4 , u_1(b) = 0 , u_1(c) = 0 ; u_2(a) = 0 , u_2(b) = 0.7 , u_2(c) = 0 and u_3(a) = 0.4 , u_3(b) = 0.7 , u_3(c) = 0. Then ω(T) = $\{ u_1, u_2, u_3, 0, 1 \}$ and $(X, \omega(T))$ is an fts. Again, let G = { a, b }. Then clearly G is countably compact in (X, T). 1_G is not countably compact fuzzy set in $(X, \omega(T))$ as there do not exist $u_k \in \{ \omega(T) \}$ (k = 1, 2, 3) such that $1_G \subseteq \bigcup_{k=1}^{3} u_k$.

Acknowledgement: The authors are grateful to the reviewers for their valuable suggestions

REFERENCES

1. D. M. Ali, Ph.D. Thesis, Banaras Hindu University; 1990.
2. C. L. Chang, Fuzzy topological spaces, *J. Math. Anal. Appl.*, 24(1968), 182 – 190.
3. David. H. Foster, Fuzzy topological groups, *J. Math. Anal. Appl.*, 67(1979), 549-564.
4. I. M. Hanafy, Fuzzy β – compactness and fuzzy β – closed spaces, *Turk J Math*, 28 (2004), 281 – 293.
5. T. E. Gantner, R. C. Steinlage and R. H. Warren, Compactness in fuzzy topological spaces, *J. Math. Anal. Appl.*, 62 (1978), 547 – 562.
6. S. Lipschutz, Theory and problems of general topology, Schaum's Outline Series, McGraw-Hill, Singapore, 1965.
7. S. R. Malghan and S. S. Benchali, On fuzzy topological spaces, *Glasnik Mat.*, 16 (36) (1981), 313 – 325.
8. Pu.Pao Ming and Liu Ying Ming, Fuzzy topology I. Neighborhood structure of a fuzzy point and Moore – Smith convergence, *J. Math. Anal. Appl.*,76 (1980), 571- 599.
9. R. Srivastava, S. N. Lal, and A K. Srivastava, Fuzzy T_1 topological Spaces, *J. Math. Anal. Appl.*, 102 (1984), 442 – 448.
10. C. K. Wong, Covering properties of fuzzy topological spaces, *J. Math. Anal. Appl.*, 43 (1973), 697 – 704.
11. L. A. Zadeh, Fuzzy Sets, *Information and Control*, 8(1965), 338 – 353.