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# A Common Fixed Point Theorem for Reciprocal Continuous Compatible Mappings in Metric Space

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***Abstract.*** In this paper, we obtain a common fixed point theorem for reciprocally continuous compatible self- mappings in a complete metric space which generalizes and improves other similar results in literature.

***Keywords:*** Metric space, compatible mappings, reciprocally continuous mappings, common fixed point

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## 1. Introduction

The fixed point theory is an important area of non-linear functional analysis. The study of common fixed point of mappings satisfying contractive type conditions has been a very active field of research during the last three decades. In 1986, Jungck [1] introduced the notion of compatible mappings, which is more general than commuting and weakly commuting mappings introduced by Sessa [12]. In 1993,Jungck et al. [2] gave a generalization of compatible mappings called compatible mappings of type (A) which is equivalent to the concept of compatible mappings under some conditions. In 1995, Pathak and Khan [3] introduced the concept of compatible mappings of type (B) with some examples to show that compatible mappings of type (B) need not be compatible of type (A). In 1998, Pathak et al. [4] introduced another extension of compatible mappings of type (A) called compatible mappings of type (C).

In 1999, Pant [11] introduced the concept of reciprocally continuous mappings and obtained common fixed point theorem.The purpose of this paper is to establish a common fixed point theorem involving compatible and reciprocally continuous mappings in a complete metric space with example.

We start with the following.

**Definition 1.1. [1]** The mappings A and S of a metric space (X, d) are said to be *compatible* iff $\lim_{n\to\infty} d(ASx_n, SAx_n) = 0$ whenever $\{x_n\}$ is a sequence in X such that $\lim_{n\to\infty} Ax_n = \lim_{n\to\infty} Sx_n = t$ for some $t \in X$.

**Definition 1.2. [2]** The mappings A and S of a metric space (X, d) are said to be *compatible of type (A)* if $\lim_{n\to\infty} d(ASx_n, SSx_n) = 0$ and $\lim_{n\to\infty} d(SAx_n, AAx_n) = 0$ whenever $\{x_n\}$ is a sequence in X such that $\lim_{n\to\infty} Ax_n = \lim_{n\to\infty} Sx_n = t$ for some $t \in X$.

**Definition 1.3. [3]** The mappings A and S of a metric space (X, d) are said to be *compatible of type (B)* if $\lim_{n\to\infty} d(ASx_n, SSx_n) \le \frac{1}{2}(\lim_{n\to\infty} d(ASx_n, At) + \lim_{n\to\infty} d(At, AAx_n))$, and $\lim_{n\to\infty} d(SAx_n, AAx_n) \le \frac{1}{2}(\lim_{n\to\infty} d(SAx_n, St) + \lim_{n\to\infty} d(St, SSx_n))$, whenever $\{x_n\}$ is a sequence in X such that $\lim_{n\to\infty} Ax_n = \lim_{n\to\infty} Sx_n = t$ for some $t \in X$.

**Definition 1.4. [4]** The mappings A and S of a metric space (X, d) are said to be compatible of type (C), if
$\lim_{n\to\infty} d(ASx_n, SSx_n) \le \frac{1}{3}[\lim_{n\to\infty} d(ASx_n, At) + \lim_{n\to\infty} d(At, AAx_n) + \lim_{n\to\infty} d(At, SSx_n)]$,
and $\lim_{n\to\infty} d(SAx_n, AAx_n) \le \frac{1}{3}[\lim_{n\to\infty} d(SAx_n, St) + \lim_{n\to\infty} d(St, SSx_n)$
$+ \lim_{n\to\infty} d(St, AAx_n)]$, whenever $\{x_n\}$ is a sequence in X such that $\lim_{n\to\infty} Ax_n = \lim_{n\to\infty} Sx_n = t$ for some $t \in X$.

**Definition 1.5. [11]** The maps A and S of a metric space (X, d) are said to be *reciprocally continuous* if $\lim_{n\to\infty} ASx_n = A(t)$ and $\lim_{n\to\infty} SAx_n = S(t)$, whenever $\{x_n\}$ is a sequence in X such that $\lim_{n\to\infty} Ax_n = t$ and $\lim_{n\to\infty} Sx_n = t$, for some $t \in X$.

It is noted that compatible, compatible of type (A), compatible of type (B) and compatible of type (C) are all equivalent if A and S are assumed to be continuous. Also, the reciprocally continuous mappings need not be continuous [10].

We need following proposition and lemma to establish our main result.

**Proposition 1.6.** If A and S are compatible and reciprocal continuous mappings on a metric space (X, d).Then, wehave

(i) $A(t) = S(t)$ where $\{x_n\}$ is a sequence in X such that $\lim_{n\to\infty} Ax_n = \lim_{n\to\infty} Sx_n = t$ for some $t \in X$

(ii) If these exist $u \in X$ such that $Au = Su = t$, then $ASu = SAu$.

**Proof:** (i) By definition of compatible mappings, we have
$\lim_{n\to\infty} SAx_n = \lim_{n\to\infty} ASx_n$.
Also, by definition of reciprocal continuous, we havelim $\lim_{n\to\infty} ASx_n = A(t)$ and $\lim_{n\to\infty} SAx_n = S(t)$.So, we get $A(t) = S(t)$.

(ii) Suppose $Au = Su = t$ for some $u \in X$. Then, we have $ASu = At$ and $SAu = St$. But, from (i), we get $At = St$. So, we have $ASu = SAu$.

**Lemma1.7. [**5] Let A,B,S and T be self-mapping of metric space (X, d) such that

$AX\subset TX, BX\subset SX$. Also, assume further that given $\varepsilon > 0$, there exists $\delta > 0$ such that for all x, y in X, we have $\varepsilon < M(x, y) < \varepsilon + \delta \Longrightarrow d(Ax, By) \leq \varepsilon$, and $d(Ax, By) < M(x, y)$ whenever $M(x, y)>0$ where $M(x,y) = \max\{d(Sx, Ty), d(Ax, Sx), d(By, Ty), [d(Sx, By) + d(Ax, Ty)]/2\}$.
Then, for each $x_0 \in X$, the sequence $\{y_n\}$ in X defined by the rule
$y_{2n} = Ax_{2n} = Tx_{2n+1}; y_{2n+1} = Bx_{2n+1} = Sx_{2n+2}$ is a Cauchy sequence.

## 2. Main Result

**Theorem 2.1.** Let (A, S) and (B, T) be compatible and reciprocally continuous pairs of self-mappings in a complete metric space (X,d) such that

(i) $AX \subset TX$ and $BX \subset SX$,

(ii) Given $\varepsilon > 0$, there exists $\delta > 0$ such that for all $x, y \in X$, we have $\varepsilon \leq M(x, y) < \varepsilon + \delta \Longrightarrow d(Ax, By) < \varepsilon$, and $d(Ax, By) < M(x, y)$ where $M(x, y) = \max\{d(Sx, Ty), d(Ax, Sx), d(By, Ty), [d(Sx, By)+d(Ax, Ty)]/2\}$, and

(iii) $d(Ax, By) < \max\{k_1[d(Sx, Ty)+ d(Ax, Sx)+ d(By, Ty)], k_2[d(Sx, By)+d(Ax, Ty)]/2\}$ for $0 \leq k_1, k_2 \leq 1$.

Then, the mappings A,B,S and T have unique common fixed point.

**Proof:** Let $x_0$ be any point in X. Define sequences $\{x_n\}$ and $\{y_n\}$ in X given by the rule
$y_{2n} = Ax_{2n} = Tx_{2n+1}; y_{2n+1} = Bx_{2n+1} = Sx_{2n+2}$ (2.1)

This can be down from (i).Then, by lemma (1.7), $\{y_n\}$ is a Cauchy Sequence. Also, since X is complete, so there exists a point z in X such that $y_n \longrightarrow z$.Now, from (1), we get $y_{2n} = Ax_{2n} = Tx_{2n+1} \rightarrow z$, $y_{n+1} = Bx_{2n+1} = Sx_{2n+2} \longrightarrow z$. (2.2)
Since (A, S) is compatible andreciprocal continuous, using the proposition (1.6), we have
$Az = Sz$. (2.3)

We claim that $Az = z$. If $Az \neq z$, then, from (iii), we get $d(Az,Bx_{2n+1}) < \max\{k_1[d(Sz,Tx_{2n+1})+d(Az,Sz)+d(Bx_{2n+1},Tx_{2n+1})], k_2[d(Sz, Bx_{2n+1})+d(Az,Tx_{2n+1})]/2\}$.

Letting $n\to\infty$, we get $d(Az, z) < d(Az, z)$ , a contradiction. Hence, we get $Az = z$. Therefore, we have $Az = Sz = z$.

Hence z be the common fixed point of A and S. Also, since $AX \subset TX$ there exists a point w in X such that $Az = Tw$. We claim that $Bw = Tw$. If $Bw \neq Tw$ from (iii), we get $d(Tw, Bw) = d (Az, Bw) < \max\{k_1[d(Sz,Tw)+d(Az, Sz)+d(Bw, Tw)], k_2[d(Sz, Bw)+d(Az,Tw)]/2\}$ which implies $d(Az, Bw) < d(Az,Bw)$, a contradiction. So, we get $Tw = Bw$.
Hence, $Az = Sz = Tw = Bw = z$. (2.4)
Now, using the proposition (1.6), we get $BTw = TBw$. (2.5)
Moreover, we get
$BBw = BTw$ and $TTw = TBw$. Hence, $BBw = BTw = TTw = TBw$. (2.6)
Again, we claim $BBw = Bw$. If $BBw \neq Bw$. Then, from (ii), we get
$d(Bw,BBw) = (Az, BBw) < \max\{d(Sz, TBw), d(Az, Sz), d(BBw, TBw), [d(Sz, BBw)+(Az, TBw)]/2\} = \max\{d(Az, BBw),0,0,d(Az, BBw)\}$,
which implies $d(Az, BBw) < d(Az, BBw)$. This is a contradiction. Hence, we have $Bw = BBw$. Now, from relation (2.6), we get $BBw = TBw = Bw$. So, we have $Bw = Az = z$.

Moreover, BBw = TBw implies that Bz = Tz = z which is the common fixed point of B and T.

Hence z is the common fixed point of A, B, S and T.

For uniqueness of the common fixed point,let $u \neq z$ is another fixed point. Then, we get Au = Su = Bu = Tu. Finally, using relation (iii), we get

$d(Az, Bu) < \max\{k_1[d(Sz, Tu)+ d(Az, Sz)+ d(Bu, Tu)], k_2[d(Sz, Bu)+d(Az, Tu)]/2\}$.

This implies d(Az, Bu) < d(Az, Bu) which is a contradiction. Hence, thecommon fixed point of A, B, S and T is unique.

This establishes the theorem.

**Example 2.2.** Let X = [2,10] and d the Euclidean metric on X. Define A, B, S and T : $X \to X$ as follows

Ax = 2 for all x,

Bx = 2 if x < 4 and $\geq 5$, Bx = 3+x if $4 \leq x < 5$,

Sx = x if $x \leq 8$, Sx = 8 if x > 8, and

Tx = 2 if x < 4 or $\geq 5$, Tx = 5 +x if $4 \leq x < 5$.

Then A, B, S and T satisfy all the conditions of the above theorem and have a unique common fixed point x = 2.

If we take A = B and S = T. Then, the Theorem 2.1 reduces to following corollary.

**Corollary 2.3.** Let (A, S) be a reciprocal continuous and compatible self-mappings of a complete metric space(X, d) such that

(i) $AX \subset SX$,

(ii) Given $\varepsilon>0$, there exists $\delta > 0$ such that for all $x, y \in X$, we have $\varepsilon \leq M(x, y) < \varepsilon + \delta \Longrightarrow d(Ax, Ay) < \varepsilon$ with $d(Ax, Ay) < M(x, y)$ where $M(x, y) = \max\{d(Sx, Sy), d(Ax, Sx), d(ay, Sy), [d(Sx, Ay)+d(Ax, Sy)]/2\}$, and

(iii) $d(Ax, Ay) < \max\{k_1[d(Sx, Sy)+ d(Ax, Sx)+ d(Ay, Sy)], k_2[d(Sx,Ay)+d(Ax, y)]/2\}$ for $0 \leq k_1, k_2 \leq 1$.

Then, A and S have a unique common fixed point.

**Remarks.** The main theorem remains true for compatible of type (A), compatible of type (B) and compatible of type (C) in place of compatible if A, S, B and T are assumed to be continuous. Our result improves the result of Jha *et al.* [6, 7], Pant and Jha [8, 9], Pant [11] and other similar results in literature.

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