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On Eccentric Graphs of Broom Graphs

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Abstract. A broom graph $B_{n,d}$ is a specific kind of graph on n vertices, having a path P with d vertices and n-d pendant vertices, all of these being adjacent to either the origin u or the terminus v of the path P. Here we consider the eccentric graph of $B_{n,d}$ and obtain its structure, besides deriving its eccentric connectivity index and domination number.

Keywords: Eccentricity, Eccentric graph, Eccentric connectivity index, Broom graph.

AMS Mathematics Subject Classification (2010): 05C12

1. Introduction

There are many special classes of graphs with very many interesting properties. One such specific kind of graphs is the class of broom graphs (see for example [4, 6, 8, 9]) which are in fact one of the types of chemical trees [7]. On the other hand, based on the distance kind of property, the notion of an eccentric graph of a given graph has been considered and its properties have been investigated [1]. The eccentric graphs are indeed underlying graphs of eccentric digraphs which have also been well-investigated [3, 5]. Here we consider the eccentric graph of a broom graph and obtain its structure. Also, we compute the eccentric connectivity index and the domination number of the eccentric graph of a broom graph.

We consider graphs which are only undirected. For unexplained notions and notations, we refer to [1, 4, 13]. A path P in a graph G is a sequence of vertices $u_1, u_2, \ldots, u_n$ such that u_i is adjacent to u_{i+1} for all i, $1 \le i \le n-1$. The distance $d(u,v)$ between two vertices u and v in a graph G is the length of the shortest path between u and v. The eccentricity $e(u)$ of a vertex u in G, with vertex set $V(G)$, is defined as $e(u) = \max_{v \in V(G)} d(u,v)$. A vertex v is an eccentric vertex of a vertex u if $d(u,v) = e(u)$. The set of all eccentric vertices of u is denoted by $E(u)$. A vertex v is an eccentric vertex of the graph G if v is an eccentric vertex of some vertex of G. The set of all eccentric vertices of G is denoted by $EP(G)$. The eccentric connectivity index [6, 8, 9] of a graph G is defined as $\xi^c(G) = \sum_{v \in V(G)} \deg(v) e(v)$. The diameter of G is defined by $diam(G) = \max_{u \in V(G)} e(u)$. A vertex u is a peripheral vertex of G if $e(u) = diam(G)$. The set of all peripheral vertices is denoted by $P(G)$. A subset $S \subseteq V(G)$ is said to be a dominating set of G, if every vertex $u \in V(G) - S$ has at least one neighbor in S. The domination number γ(G) of a graph G is the cardinality of a minimal dominating set of G. A complete graph K_n on n vertices has an edge between every pair of distinct vertices. The complement of a graph G has the same vertex set as G and two vertices u and v are adjacent in G if and only if they are non-adjacent in G. A graph G is called a k-partite graph if it is partitioned into k distinct sets and no edge has both ends in the same partition.

The union of the graphs $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ is the graph $G_1 \cup G_2 = (V_1 \cup V_2, E_1 \cup E_2)$ and the sum $G_1 + G_2$ is the graph $G_1 \cup G_2$ with each vertex of G_1 joined to every vertex of G_2. For three or more graphs $G_1, G_2, \ldots, G_n$, the sequential join $G_1 + G_2 + G_3 + \ldots + G_n$ is the graph $(G_1 + G_2) \cup (G_2 + G_3) \cup (G_3 + G_4) \cup \ldots \cup (G_{n-1} + G_n)$. The sequential join of $K_3 + K_2 + K_3$ is shown in Fig. 1.

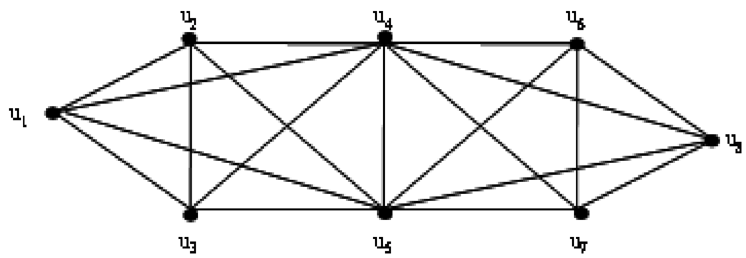

Figure 1: K_3+K_2+K_3

The broom graph [8, 9] $B_{n,d}$ consists of a path P with d vertices, together with (n-d) pendant vertices all adjacent to the same end vertex of P. The eccentric graph G_e [1] of a graph G is defined as a graph having the same set of vertices as G with two vertices u and v adjacent in G_e if and only if either u is an eccentric vertex of v or v is an eccentric vertex of u. A broom graph $B_{9,5}$ and its eccentric graph are respectively shown in Fig.2 and Fig. 3.

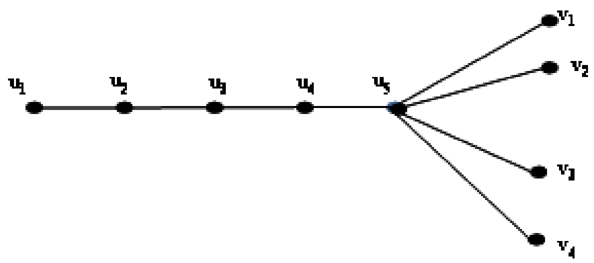

Figure 2: Broom graph $B_{9,5}$

Another broom graph $B_{9,6}$ and its eccentric graph are shown in Fig.4 and Fig. 5.

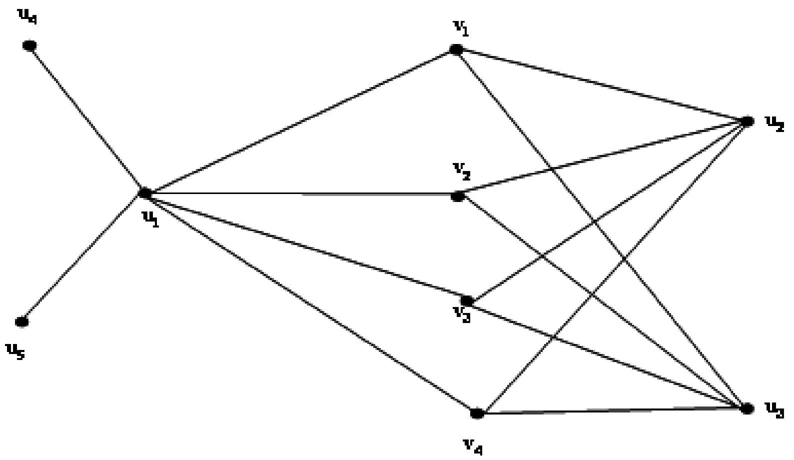

Figure 3: Eccentric graph of $B_{9,5}$

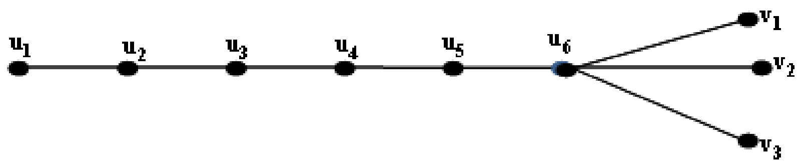

Figure 4: Broom graph $B_{9,6}$

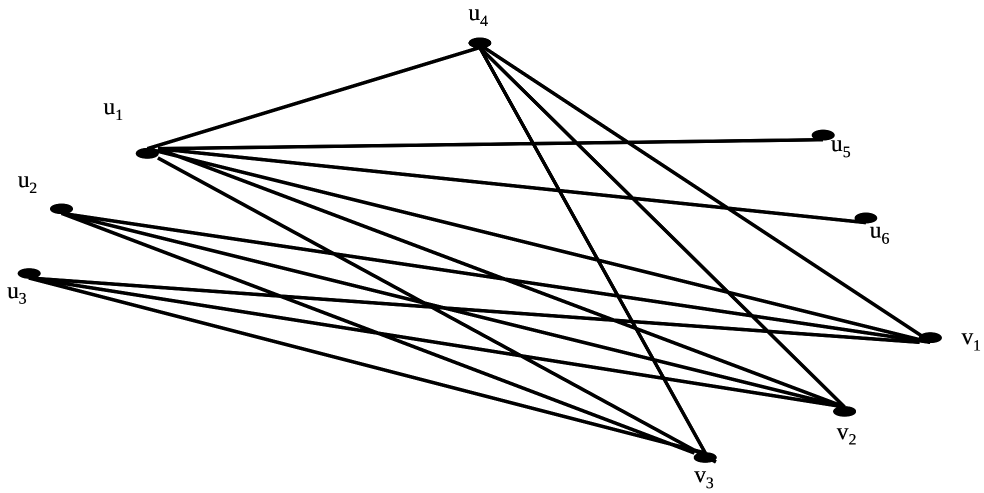

Figure 5: Eccentric graph of $B_{9,6}$

2. Eccentric Graph of a Broom graph

In this section, we derive in Theorem 2.1, the structure of the eccentric graph of a broom graph. We also obtain the domination numbers of a broom graph and its eccentric graph in Theorems 2.2 and 2.3. The eccentric connectivity index of a broom graph is known [9]. Here we obtain this in Theorem 2.5, for the eccentric graph of a broom graph.

Theorem 2.1. Given the broom graph $G = B_{n,d}$, the eccentric graph G_e is

(i) $\overline{K_{\left(\frac{d-1}{2}\right)}} + K_1 + \overline{K_{n-d}} + \overline{K_{\left(\frac{d-1}{2}\right)}}$, when d is odd

(ii) a 3-partite graph., when d is even.

Proof. Let $G = B_{n,d} = (V,E)$ be a broom graph. Let $V(G) = \{ u_1, u_2, u_3,\ldots, u_d, v_1, v_2, v_3, \ldots, v_{n-d}\}$ such that $u_1u_2 u_3\ldots u_d$ is a path on d vertices and $v_1, v_2, v_3,\ldots, v_{n-d}$ are pendant vertices that are adjacent to u_d . Then $P(G) = EP(G) = \{u_1, v_1, v_2, v_3, \ldots, v_{n-d}\}$. In order to prove (*i*) , let d be odd. Then

$E(u_i) = \{ v_1, v_2, v_3,\ldots, v_{n-d}\}$, for $1 \le i \le \frac{d+1}{2}$

$E(u_i) = \{ u_1\} = E(v_j)$ for $i > \frac{d+1}{2}$ and $1 \le j \le n-d$. This implies that the eccentric graph G_e has the same vertex set as G and in G_e , every vertex u_i, $i > \frac{d+1}{2}$ is adjacent to u_1, the vertex u_1 is adjacent to all v_j's, $1 \le j \le n-d$, that are adjacent to u_i's, $1 \le i \le \frac{d+1}{2}$. Therefore G_e is of the form

$$\overline{K_{\frac{d-1}{2}}} + K_1 + \overline{K_{n-d}} + \overline{K_{\frac{d-1}{2}}} .$$

In order to prove (*ii*) , let d be even. Then $E(u_i) = \{ v_1, v_2, v_3,\ldots, v_{n-d}\}$, for $1 \le i < \frac{d}{2}+1$

$E(u_i) = \{ u_1\} = E(v_j)$ for $i > \frac{d}{2}+1$ and $1 \le j \le n-d$

and $E(u_{(d/2+1)}) = \{ u_1, v_1, v_2, v_3,\ldots, v_{n-d}\}$. This implies that the eccentric graph G_e has the same vertex set as G and every vertex u_i, $1 \le i < \frac{d}{2}+1$, is adjacent to all v_j's ,

$1 \le j \le n-d$, every vertex u_i, $i > \frac{d}{2}+1$ is adjacent to u_1 and the vertex $u_{\left(\frac{d}{2}+1\right)}$ is adjacent to u_1 and all v_j's , $1 \le j \le n-d$. In other words, the vertex set of G_e is partitioned into 3 sets, $\left\{u_{\left(\frac{d}{2}+1\right)}\right\}$, $\left\{u_1, u_2, u_3, \ldots, u_{\left(\frac{d}{2}\right)}\right\}$ and

$\left\{u_{\left(\frac{d}{2}+1\right)}, u_{\left(\frac{d}{2}+3\right)}, \ldots, u_d, v_1, v_2, \ldots, v_{(n-d)}\right\}$ such that no edge in G_e has both ends in the same partition. Therefore G_e is a 3-partite graph.

Theorem 2.2. $\gamma(B_{n,d}) = \begin{cases} \dfrac{d+1}{2}, & if\ d\ is\ odd \\ \dfrac{d}{2}, & if\ d\ is\ even \end{cases}$

Proof. Let $G = B_{n,d} = (V,E)$ be a broom graph with $V(G)$ is as in proof of Theorem 2.1. It is clear that when d is respectively odd and even, the sets $\{u_1, u_3, u_5, \ldots, u_d\}$ and $\{u_2, u_4, u_6, \ldots, u_d\}$ are minimal dominating sets of G. Thus we have the domination number as given in the statement of the theorem.

Theorem 2.3. If $G = B_{n,d}$ is a broom graph, then $\gamma(G_e) = 2$.

Proof. Let $G = B_{n,d} = (V,E)$ be a broom graph with $V(G)$ is as in proof of Theorem 2.1. Now the eccentric graph G_e has the structure as given in Theorem 2.1. So whether d is odd or even, the set consisting of u_1 and any one of v_j's is a minimal dominating set. Therefore $\gamma(G_e) = 2$.

Theorem 2.4. [9] The eccentric connectivity index of a broom graph $B_{n,d}$ is

$$\xi^c(B_{n,d}) = \begin{cases} 2dn - n - \left(\dfrac{d^2}{2}\right) - d + 1, & when\ d\ is\ even \\ \dfrac{1}{2}\left(3 - 2d - d^2 - 2n + 4dn\right), & when\ d\ is\ odd \end{cases}$$

Theorem 2.4. If G is a broom graph $B_{n,d}$ then

$$\xi^c(G_e) = \begin{cases} \dfrac{5}{2}nd - \dfrac{5}{2}d^2 + \dfrac{3}{2}n + d - \dfrac{5}{2}, & when\ d\ is\ odd \\ \dfrac{5}{2}nd + 3n - \dfrac{d}{2} - \dfrac{5}{2}d^2 - 1 & when\ d\ is\ even \end{cases}$$

Proof. Let $G = B_{n,d} = (V,E)$ be a broom graph with $V(G)$ is as in proof of Theorem 2.1. If d is odd, then by Theorem 2.1, the eccentric graph G_e is $\overline{K_{\left(\frac{d-1}{2}\right)}} + K_1 + \overline{K_{n-d}} + \overline{K_{\left(\frac{d-1}{2}\right)}}$ and every vertex in the set $\left\{u_{\left(\frac{d+3}{2}\right)}, u_{\left(\frac{d+5}{2}\right)}, \ldots, u_d\right\}$ is adjacent to u_1, every vertex in the set $\left\{u_2, u_3, \ldots, u_{\left(\frac{d+1}{2}\right)}\right\}$ is adjacent to all v_j's, $1 \le j \le n-d$ and all v_j's, $1 \le j \le n-d$, are adjacent to u_1. This implies that in G_e, $e(u_1) = e(v_j) = 2$, $1 \le j \le n-d$ and $e(u_i) = 3$, $2 \le i \le d$. Also, $deg(u_i) = 1$, $\frac{d+3}{2} \le i \le d$, $deg(u_i) = n-d$, $2 \le i \le \frac{d+1}{2}$, $deg(u_1) = n - \frac{(d+1)}{2}$ and $deg(v_j) = \frac{d+1}{2}$, $1 \le j \le n-d$.

This implies that to the eccentric connectivity index of G_e , the vertex u_1 contributes *2n-(d+1)*, each vertex u_i, $2 \le i \le \frac{d+1}{2}$ contributes *3(n-d)*, each vertex u_i, $\frac{d+3}{2} \le i \le d$, contributes 3 and each vertex v_j , $1 \le j \le n-d$, contributes d+1. On summing up, we obtain

$$\xi^c(G_e) = 2n-(d+1) + 3\frac{(d-1)(n-d)}{2} + 3\frac{(d-1)}{2} + (n-d)(d+1)$$

$$= \frac{5}{2}nd - \frac{5}{2}d^2 + \frac{3}{2}n + d - \frac{5}{2}$$

If d is even , then by Theorem 2.1, the eccentric graph G_e is a 3-partite graph with partitions $\left\{ u_{\left(\frac{d}{2}+1\right)} \right\}$, $\left\{ u_1, u_2, u_3, \ldots, u_{\left(\frac{d}{2}\right)} \right\}$ and $\left\{ u_{\left(\frac{d}{2}+2\right)}, u_{\left(\frac{d}{2}+3\right)}, \ldots, u_d, v_1, v_2, \ldots, v_{(n-d)} \right\}$ such that $u_{\left(\frac{d}{2}+1\right)}$ is adjacent to u_1 as well as all v_j's, $1 \le j \le n-d$ and each vertex u_i, $1 \le i \le \frac{d}{2}$ is adjacent to all v_j's, $1 \le j \le n-d$ and each vertex u_i, $\frac{d}{2}+2 \le i \le d$, is adjacent to u_1. This implies that in G_e, $e(u_1) = e(v_j) = e\left(u_{\left(\frac{d}{2}+1\right)}\right)$ =2, $1 \le j \le n-d$ and $e(u_i)$ =3, $i = 2,3,\ldots,\frac{d}{2},\frac{d}{2}+2,\ldots,d$. Also, $deg(u_i)$ =1, $\frac{d}{2}+2 \le i \le d$, $deg(u_i)$ =*n-d*, $2 \le i \le \frac{d}{2}$

$\deg\left(u_{\left(\frac{d}{2}+1\right)}\right) = n-d+1$, $deg(u_1) = n - \frac{d}{2}$ and $deg(v_j) = \frac{d}{2}+1$, $1 \le j \le n-d$.

Hence the vertex u_1 contributes *2n-d*, each vertex u_i, $2 \le i \le \frac{d}{2}$, contributes *3n-3d,* each vertex u_i, $\frac{d}{2}+2 \le i \le d$, contributes 3, the vertex $u_{\left(\frac{d}{2}+1\right)}$ contributes *2n -2d +2* and each vertex v_j , $1 \le j \le n-d$, contributes $2\left(\frac{d}{2}+1\right)$ to the eccentric connectivity index of G_e. On summing up, we obtain

$$\xi^c(G_e) = 2n - d + 3\left(\frac{d}{2}-1\right)(n-d) + 2(n-d+1) + 3\left(\frac{d}{2}-1\right) + 2(n-d)\left(\frac{d}{2}+1\right)$$

$$= \frac{5}{2}nd + 3n - \frac{d}{2} - \frac{5}{2}d^2 - 1$$

3. Conclusion

Eccentric graphs of other special kinds of graphs such as intersection graphs [10], trapezoid graphs [2] and so on can be investigated. It will also be of interest to examine the role of eccentric graphs of the type considered here in graph based probabilistic models such as Bayesian networks that have wide applications in biometrics, medical image analysis and so on, as for example, the Bayesian network based frame work to recognize complex faces proposed in [12].

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