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Fixed Point Theorems for Occasionally Weakly Compatible Mappings in Semi-Metric Space

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Abstract. In this paper, we establish a common fixed point theorem for three pairs of self mappings in semi-metric space for occasionally weakly compatible mappings which improves and extends similar known results in the literature.

Keywords: Semi-metric space, occasionally weakly compatible mappings, common fixed point

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1. Introduction

The fixed point theory has become a part of non-linear functional analysis since 1960. It serves as an essential tool for various branches of mathematical analysis and its applications. Polish mathematician Banach published his contraction Principle in1922. In 1928, Menger[17] introduced semi-metric space as a generalization of metric space. In 1976, Cicchese [6] introduced the notion of a contractive mapping in semi-metric space and proved the first fixed point theorem for this class of spaces. In 1986, Jungck [13] introduced the notion of compatible mappings. In 1997, Hicks and Rhoades[8] generalized Banach contraction principle in semi-metric space. In 1998, Jungck and Rhoades [14] introduced the notion of weakly compatible mappings and showed that compatible mappings are weakly compatible but not conversely. Recently in 2006,Jungck and Rhoades [15] introduced occasionally weakly compatible mappings which is more general among the commutativity concepts. Jungck and Rhoades[15] obtained several common fixed point theorems using the idea of occasionally weakly compatible mappings. Several interesting and elegant results have been obtained by various authors in this direction. There have been interesting generalized and formulated results in semi-metric space initiated by Frechet [7], Menger [17] and Wilson[20]. Also, in this paper, we prove a common fixed point theorem for three pairs of self-mappings using occasionally weakly compatible mappings.

Let X be a non-empty set and $d: X \times X \to [0, \infty)$. Then, (X, d) is said to be a **semi-metric space** (symmetric space) if and only if it satisfies the following:

W1 : $d(x,y) = 0$ if and only if $x = y$, and

W2 : $d(x,y) = d(y,x)$ if and only if $x = y$ for any $x, y \in X$.

The difference of a semi-metric and a metric comes from the triangle inequality.

Definition 1.1. [1] Let A and B be two self-mappings of a semi-metric space (X,d). Then, A and B are said to be **compatible** if $\lim_{n\to\infty} d(ABx_n, BAx_n) = 0$, whenever $\{x_n\}$ is a sequence in X such that $\lim_{n\to\infty} d(Ax_n, t) = \lim_{n\to\infty} d(Bx_n, t) = 0$, for some $t \in X$.

Definition 1.3. [1] Let A and B be two self-mappings of a semi-metric space (X,d). Then, A and B are said to be **weakly compatible** if they commute at their coincidence points.

Definition 1.4. [15] Let A and B be two self-mappings of a semi-metric space (X,d). Then, A and B are said to be **occasionally weakly compatible (owc)** if there is a point $x \in X$ which is coincidence point of *A* and *B* at which *A* and *B* commute.

Example 1.1. Let us consider $X = [2, 20]$ with the semi-metric space (X,d) defined by $d(x,y) = (x-y)^2$. Define a self map$A$ and B by

$A(2) = 2$at$x = 2$ and $A(x) = 6$ for $x > 2$

$B(2) = 2$at $= 2$, $B(x) = 12$ for $2 < x \le 5$ and $B(x) = x - 3$ for $x > 5$.

Now, $A(9) = B(9) = 6$, besides $x = 2$, $x = 9$ is another coincidence point of *A* and *B*. $AB(2) = BA(2)$but$(9) = 6$ $BA(9) = 3$, $AB(9) \neq BA(9)$. Therefore *A* and *B*are owc but not weakly compatible. Hence weakly compatible mappings are owc but not conversely.

Lemma 1.1. [15] Let(X,d)be a semi-metric space. If the self mappings$A$ and B on X have a unique point of coincidence$w = Ax = Bx$, then w is the unique common fixed point of *A* and *B*.

In order to establish our result, we consider a function $\emptyset: \mathbb{R}^+ \to \mathbb{R}^+$ satisfying $(\emptyset 1) 0 < \emptyset(t) < t$, for $t > 0$, and $(\emptyset 2)$ for each $t > 0$, $\lim_{n\to\infty} \emptyset^n(t) = 0$.

2. Main Results

Theorem 2.1. Let (X,d) be a semi-metric space. Let *A*, *B*, *T*, *S*, *P* and *Q* be self-mappings of *X* such that

(i) $\{AB, P\}$ and $\{TS, Q\}$ are occasionally weakly compatible (owc),

(ii) $d(ABx, TSy) \le \emptyset(max\ \{d(Px,Qy), \frac{1}{2}[d(ABx,Px) + d(TSy,Qy)], \frac{1}{2}[d(ABx,Qy) + d(TSy,Px)]\}\ )$for all $(x,y) \in X \times X$,

Then *AB*, *TS*,*P* and *Q* have a unique common fixed point. Furthermore, if the pairs (*A*, *B*) and (*T*, S) are commuting pair of mappings then *A*, *B*, *T*, *S*, *P* and *Q* have a unique common fixed point.

Proof: Since $\{AB, P\}$ and $\{TS, Q\}$ are owc, then there exists $x, y \in X$ such that $ABx = Px$and $TSy = Qy$. We claim that $ABx = TSy$. Using condition (ii), we get

$d(ABx, TSy) \leq \emptyset(max\,\{d(Px, Qy), \frac{1}{2}[d(ABx, Px) + d(TSy, Qy)], \frac{1}{2}[d(ABx, Qy) + d(TSy, Px)]\})$
$= \emptyset(max\,\{d(ABx, TSy), \frac{1}{2}[d(ABx, ABx) + d(TSy, TSy)], \frac{1}{2}[d(ABx, TSy) + d(TSy, ABx)]\})$
$= \emptyset(max\,\{d(ABx, TSy), 0, d(ABx, TSy)\})$
$= \emptyset(max\,\{d(ABx, TSy)\})$
$= \emptyset(\, d(ABx, TSy)\,)$
$< d(ABx, TSy)$
which is contradiction. So, $ABx = TSy$. Therefore, $ABx = Px = TSy = Qy$. (2.1)

Moreover, if there is another point of coincidence z such that $ABz = Pz$, then usingcondition (ii), we get

$ABz = Pz = TSy = Qy$(2.2)

Also from (2.1) and (2.2), it follows that $ABz = ABx$. This implies that $z = x$. Hence, $w = ABx = Px$, for $w \in X$, is the unique point of coincidence of $AB$and P. By Lemma 1.1, w is the unique common fixed point of $AB$and P. Hence $ABw = Pw = w$. Similarly, there is a unique common fixed point $u \in X$ such that $u = TSu = Qu$. Suppose that $u \neq w$.Then using condition (ii), we get.

$$
\begin{aligned}
d(w, u) &= d(ABw, TSu) \\
&\leq \emptyset(max\,\{d(Pw, Qu), \frac{1}{2}[d(ABw, Pw) + d(TSu, Qu)], \frac{1}{2}[d(ABw, Qu) \\
&\qquad + d(TSu, Pw)]\}) \\
&= \emptyset(max\,\{d(w, u), \frac{1}{2}[d(w, w) + d(u, u)], \frac{1}{2}[d(w, u) + d(u, w)]\}) \\
&= \emptyset(max\,\{d(w, u), 0, d(w, u)\}) \\
&= \emptyset(d(w, u)) \\
&< d(w, u).
\end{aligned}
$$

This is contradiction. Therefore, we have $w = u$. Hence, w is the unique common fixed point of AB, TS, P and Q. Finally, we need to show that$w$ is only the common fixed point of mappings A, B, T, S, P and Q. If the pairs (A, B) and (T, S) are commuting pairs, then for this, we can write

$Aw = A(ABw) = A(BAw) = AB(Aw)$. This implies that $Aw = w$. Also, $Bw = B(ABw) = BA(Bw) = AB(Bw)$. This implies that $Bw = w$.
Similarly, we have $Tw = w$and $Sw = w$.
Hence A, B, T, S, P and Q have a unique common fixed point.

Example 2.1. Consider $X = [0, 1]$ with the semi-metric space (X, d) defined by $d(x, y) = (x - y)^2$. Define self mappings$A$, B, T, S, P and Q as $Ax = \frac{x+1}{2}$, $Bx = \frac{2+3x}{5}$, $Tx = \frac{2x+1}{3}$, $S(x) = \frac{x+3}{4} P(x) = \frac{3x+1}{4}$ and $Q(x) = \frac{2x+3}{5}$. Also, the mappings satisfy all the conditions of above Theorem 2.1 and hence have a unique common fixed point $x = 1$.

On the basis of above Theorem 2.1, we have the following corollary.

Corollary 2.1. Let (X,d) be a semi-metric space. Let A, B, T, S, P and Q be self-mappings of X such that
(i) $\{AB, P\}$ and $\{TS, Q\}$ are occasionally weakly compatible (owc),
(ii) $d(ABx, TSy) \leq \emptyset(max\,\{d(Px, Qy), d(ABx, Qy), d(TSy, Px), \frac{1}{2}[d(ABx, Px) + d(TSy, Qy)]\})$ for all $(x, y) \in X \times X$,
then AB, TS, P and Q have a unique common fixed point. Furthermore, if the pairs (A, B) and (T, S) are commuting pair of mappings then A, B, T, S, P and Q have a unique common fixed point.

In the above Theorem 2.1, if we take $A = B$ and $= S$, then we have the following corollary. This is the result of G. Jungck and B.E. Rhoades [14].

Corollary 2.2. Let (X,d) be a semi-metric space. Let A, T, P and Q be self- mappings of X such that
(i) $\{A, P\}$ and $\{T, Q\}$ are occasionally weakly compatible (owc),
(ii) $d(Ax, Ty) \leq \emptyset(max\,\{d(Px, Qy), \frac{1}{2}[d(Ax, Px) + d(Ty, Qy)], \frac{1}{2}[d(Ax, Qy) + d(Ty, Px)]\})$ for all $(x, y) \in X \times X$,
then A, T, P and Q have a unique common fixed point.

In Theorem 2.1, if we take $A = B = Q$ and $T = S = P$, then we have the following corollary.

Corollary 2.3. Let (X,d) be a semi-metric space. Let A and T be self- mappings of X such that
(i) A and $T$are occasionally weakly compatible (owc),
(ii) $d(Ax, Ty) \leq \emptyset(max\,\{d(Tx, Ay), \frac{1}{2}[d(Ax, Tx) + d(Ty, Ay)], \frac{1}{2}[d(Ax, Ay) + d(Ty, Tx)]\})$ for all $(x, y) \in X \times X$,
then A and T have a unique common fixed point.

Remarks 2.1. Our result generalizes the result of Jungck and Rhoades [15], Manro [16], Pant and Chauhan [19] and other similar results in the semi-metric space.

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REFERENCES

1. M. Aamri and D. El. Moutawakil,Common fixed points under contractive conditions in symmetric space, *Applied Mathematics E-Notes*, 3(2003) 156-162.
2. I.D.Arandelovic and D.S.Petkovic,A note on some fixed point results, *Applied Mathematics E-Notes*, 9(2009) 254-261.
3. E.W.Chittenden, On the equivalence of ecartandvoisinage, *Trans. Amer. Math. Soc*, 18(1917) 161-166.

4. S.H.Cho, G.Y .Lee and J.S.Bae, On coincidence and fixed point theorems in symmetric space, *Fixed Point Theory Appl.*, 2008 (2008), Art. ID 562130, 9 pages.
5. S.H.Choand. D.J. Kim, fixed point theorems for generalized contractive type mappings in symmetric space, *Korean J. Math Appl.,* 16 (2008) 439-450.
6. M. Cicchese, Questioni di comptetezza e contrazioni in spazimetrici generalization, *Boll. Un. Mat. Ital.,*13-A(5) (1976) 175-179.
7. M. Fr´echet, Sur quelques points du calculfonctionnel, *Rend. Circ. Mat. Palermo,* 22 (1906) 1–74.
8. T.L. Hicks and B.E.Rhoades, Fixed point theory in symmetric space with applications to probabilistic space, *Nonlinear Analysis*, 36 (1999) 331-334.
9. M. Imdad and Q.H.Khan, Six mappings satisfying a rational inequality, *Radovi Matematicki*, 9 (1999) 251-260.
10. M. Imdad, J. Ali and L.Khan, Coincidence and fixed points in semi-metric spaces under strict conditions, *J. Math. Anal. Appl*, 320 (2006) 352-360.
11. M.Imdadand and J. Ali, Common fixed point theorems in semi-metric space employing a new implicit function and common property (E.A.), *Bull. Belgium.Math. Soc. Simon Stevin,* 16 (2009) 421-433.
12. K. Jha, R.P. Pant and K.B. Manandhar, A common fixed point theorem for reciprocal continuous compatible mappings in metric space, *Annals of Pure and Appl. Maths,* 5(2) (2014) 120-124.
13. G.Jungck, Compatible mappings and common fixed point, *International J. Math. Math. Sci*, 9 (1986)771-779.
14. G. Jungck and B.E.Rhoades, Fixed point for set valued function without continuity, *Indian J. Pure Appl. Math..,* 293 (3) (1998) 227-238.
15. G. Jungck and B.E.Rhoades, Fixed point theorem for occasionally weakly compatible mappings, *Fixed Point Theory*, 7 (2006) 280-296.
16. S.Manro, Fixed point theorems for occasionally weakly compatible maps in fuzzy semi-metric space, to appear in *J. Appl. Maths. and Computations.*
17. K. Menger,Untersuchungenuberallgemeine, *Math Annalen*, 100 (1928) 75-163.
18. R.P. Pant,Common fixed point of non-commuting mappings, *J. Math Anal. Appl,* 188 (1994) 436-440.
19. B. D. Pant and S. Chauhan, Common fixed point theorem for occasionally weakly compatible mappings in menger space, *Surveys in Maths. Appl.,*6 (2011) 1-7.
20. W.A.Wilson, On semi-metric space, *Amer. J. Math,* 53 (1931) 361-373.