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Some Structural Properties of Semirings

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Abstract. In this paper, some structural properties of semirings are investigated. This is done by introducing some examples of semirings, especially a class of finite semirings. Examples and results are illustrated by computing using **MATLAB.**

Keywords: Additively inverse semiring, Idempotent semiring, Regular semiring, Boolean semiring

AMS Mathematics Subject Classification (2010): 16Y60

1. Introduction

Many researchers have studied different aspects of semiring. The notion of semiring was first introduced by Vandiver in 1934. Vandiver introduced an algebraic system, which consists of non empty set S with two binary operations addition (+) and multiplication (.). The system (S; +, .) satisfies both distributive laws but does not satisfy cancellation law of addition. The system he constructed was ring like but not exactly ring. Vandiver called this system a 'Semiring'. Luce [10] characterized the invertible matrices over a Boolean algebra of at least two elements. Rutherford [2] has introduced that a square matrix over a Boolean algebra of 2 elements is invertible. Additively inverse semirings are studied by Karvellas [9]. Kaplansky [4], Petrich [6], Goodearl [4], Reutenauer [1], Fang [5] have studied semiring.

2. Preliminaries

In this section, we present some definitions and examples of algebraic structures of semigroup and semiring. We support these definitions by some examples. In some cases works are done by MATLAB function scripts. MATLAB function scripts are not presented in this paper but input and outputs from the computer are given.

Definition 2.1. Let S be non empty set with binary operation *. Then the algebraic structure (S; *) is called ***a semigroup*** iff $\forall a,b,c \in S;\ a*(b*c) = (a*b)*c$.

Definition 2.2. Let S be non empty set with two binary operations **+** and **.** . Then the algebraic structure (S; +, .) is called **a *semiring*** iff $\forall a, b, c \in S$;

(i) (S; +) is a semigroup
(ii) (S; .) is a semigroup
(iii) a. (b+c) = a.b + a.c and (b+c).a = b.a + c.a.

Example 2.2(a). (B = {0,1}, +, .) is a semiring, where + and . are defined by

+	0	1
0	0	1
1	1	1

.	0	1
0	0	0
1	0	1

Example 2.2(b). (I =[0,1]; +, .) is a semiring, where order in [0,1] is usual $\leq$ and + and . are defined as follows:

a + b = max{a,b}, a.b = min{a,b}; $\forall a,b \in I$.

Example 2.2(c). (A class of finite semirings)

For integers $n \geq 2$ and $0 \leq i < n$, let $m = n - i$ and $N_n = \{0,1,2,.....,n-1\}$

On N_n, define

$$x +_{n,i} y = \begin{cases} x+y & if \quad x+y \leq n-1 \\ l^+ & if\ x+y \geq n \end{cases}$$

and

$$x \ *_{n,i}\ y = \begin{cases} xy & if\ \ xy \leq n-1 \\ l^* & if\ \ xy \geq n \end{cases}$$

where $i \leq l^+ \leq n-1$ and $x + y \equiv_m l^+$

$i \leq l^* \leq n-1$ and $xy \equiv_m l^*$

$+_{n,i}$ and $*_{n,i}$ are well defined binary operations on N_n, In fact if $n \geq 2$ and $0 \leq i \leq n, k \geq n, (n,i,k \in N_n)$, then by division algorithm, there exists unique r such that $0 \leq r < n-i$ and $k - i = (n-i)j + r,$ so that $i \leq r + i < n$ and $k = (n-i)j + r + i.$ which implies , taking $l = r + i$, $i \leq l < n$ and $k \equiv_{n-i} l.$ Note that $l = \text{mod}(k-i, n-i) + i$.

As addition and multiplication of integers are associative and commutative, so $(N_n, +_{n,i},\ *_{n,i})$ is a commutative semiring with zero (0) and identity (1).

We denote this semiring by $N_{n,i}$.

Two examples:

(i) $N_{n,0} = Z_n$ with usual residue addition and multiplication is a semiring.

(ii) $N_{2,1} = B$ is a semiring. It is also called Boolean semiring.

Example 2.2(d). L = {0,1, 2, 3, 4, 5, 6, 7, 8, 9} is a commutative semiring with zero, where addition and multiplication are defined as 2.2(c). Here n = 10, i = 7.

The MATLAB function scripts are not shown. Outputs are presented below.

```
>> A = [0 1 2 3 4 5 6 7 8 9]

A =
     0    1    2    3    4    5    6    7    8    9

>> ClassSemiring(A)
ans =
```

$+_{10,7}$	0	1	2	3	4	5	6	7	8	9
0	0	1	2	3	4	5	6	7	8	9
1	1	2	3	4	5	6	7	8	9	7
2	2	3	4	5	6	7	8	9	7	8
3	3	4	5	6	7	8	9	7	8	9
4	4	5	6	7	8	9	7	8	9	7
5	5	6	7	8	9	7	8	9	7	8
6	6	7	8	9	7	8	9	7	8	9
7	7	8	9	7	8	9	7	8	9	7
8	8	9	7	8	9	7	8	9	7	8
9	9	7	8	9	7	8	9	7	8	9

```
>> A=[0 1 2 3 4 5 6 7 8 9]
A =
     0    1    2    3    4    5    6    7    8    9
>> ClassSemiringMult(A)
ans =
```

$*_{10,7}$	0	1	2	3	4	5	6	7	8	9
0	0	0	0	0	0	0	0	0	0	0
1	0	1	2	3	4	5	6	7	8	9
2	0	2	4	6	8	7	9	8	7	9
3	0	3	6	9	9	9	9	9	9	9
4	0	4	8	9	7	8	9	7	8	9
5	0	5	7	9	8	7	9	8	7	9
6	0	6	9	9	9	9	9	9	9	9
7	0	7	8	9	7	8	9	7	8	9
8	0	8	7	9	8	7	9	8	7	9
9	0	9	9	9	9	9	9	9	9	9

Definition 2.3. Let (S; +, .) be a semiring**.** Then S is called

(i) ***additively commutative*** iff $\forall x, y \in S,\ x + y = y + x.$

(ii) ***multiplicatively commutative*** iff $\forall x, y \in S,\ x.y = y.x.$

(S; +, .) is called ***a commutative semiring*** iff both (i) and (ii) hold.

Example 2.3(a). $R_0^+ = \{x \in R : x \geq 0\}$**,** $Z_0^+ = \{x \in Z : x \geq 0\}$**,** $Q_0^+ = \{x \in Q : x \geq 0\}$ are commutative semirings with zero which are not rings.

Definition 2.4. Let (S; +, .) be a semiring**.** Then an element $0 \in S$ is called ***zero of S*** iff $\forall x \in S$;

$$x + 0 = x = 0 + x \text{ and } x.0 = 0 = 0.x.$$

Definition 2.5. Let (S; +, .) be a semiring**.** Then an element $1 \in S$ is called ***identity of S*** iff $\forall x \in S$;

$$x.1 = x = 1.x.$$

Proposition 2.6. Let (S; +, .) be a semiring. Then both a zero and an identity element of S (if exist) are unique.

Proof : Trivial.

Definition 2.7. Let (S; +, .) be a commutative semiring with zero**.** It is called ***Boolean semiring*** iff $\forall x \in S$;

$$x.x = x.$$

Example 2.7(a).

(i) (B = {0,1}; +, .) is a *Boolean semiring*; where " + " and " . " are defined in 2.2(a)

(ii) (I =[0,1]; +, .) is a *Boolean semiring*; where order in [0,1] is usual $\leq$ and " + " and " . " are defined as follows:

a + b = max{a,b}, a.b = min{a,b}.

(iii) Let $X \neq \phi$ and P (X) is power set of X. + and . are defined by $A + B = A \cup B$ and A.B $= A \cap B; \forall A, B \in P(X)$. Then (P (X); +, .) is a ***Boolean semiring***, where ϕ and X are zero and identity of P (X).

Definition 2.8. Let (S; +, .) be a semiring**.** Then an element $x \in S$ is called ***additively invertible*** in ***S*** iff $\forall x \in S,\ \exists! y \in S$ such that

$$x + y = 0 = y + x.$$

Definition 2.9. Let (S; +, .) be a semiring with identity**.** Then an element $x \in S$ is called ***multiplicatively invertible*** in ***S*** iff $\forall x \in S,\ \exists! y \in S$ such that

$$x.y = 1 = y.x.$$

Definition 2.10. Let (S; +,.) be a semiring. Then S is called ***regular semiring*** iff $\forall x \in S, \exists\, y \in S$ such that $xyx = x$

Definition 2.11. Suppose (S; +, .) is semiring . Then S is called ***additively inverse*** if

(S; +) is an inverse semigroup i.e. for each $x \in S, \exists! x' \in S$ such that $x = x + x' + x$ and $x' = x' + x + x'$.

Proposition 2.12. (B = {0,1}; +, .) is a regular semiring.

Proof : We have B = {0,1}. By using **.** table

.	0	1
0	0	0
1	0	1

Let us construct a table:

x	y	x. y. x
0	**0**	0.0.0 = 0
0	**1**	0.1.0 = 0
1	**0**	1.0.1 = 0
1	**1**	1.1.1 = 1

$\forall x \in B$, $\exists$ $y \in B$ such that

$$x.y.x = x.$$

Therefore (B = {0,1}; +, .) is regular semiring. Δ

Proposition 2.13. R_0^+ and Q_0^+ are regular semiring under usual addition and multiplication.

Proof : We have

$$R_0^+ \subset R.$$

For $x = 0 \in R_0^+$, for any $y \in R_0^+$ and x y x = 0.y.0 = 0 = x.

For $x = a \in R_0^+$, other than 0 we get $y = \frac{1}{a} \in R_0^+$ and

$$\text{x y x} = a.\frac{1}{a}.a = a = \text{x}.$$

$\forall x \in R_0^+$, $\exists$ $y \in R_0^+$ such that

$$x\,y\,x = x.$$

The case for Q_0^+ is similar.

Proposition 2.14. $Z_0^+ = \{x \in Z : x \ge 0\}$ $R_0^+ = \{x \in R : x \ge 0\}$ and $Q_0^+ = \{x \in Q : x \ge 0\}$ are commutative semiring with zero but not rings.

Proof : We have

$$Z_0^+ \subset Z.$$

Clearly Z_0^+ is commutative semiring with zero.

Let us show (Z_0^+ ; +, .) is not ring:

For $x = 2 \in Z_0^+$

But $y = -2 \notin Z_0^+$

So that x + y = 0

Therefore (Z_0^+ ; +, .) is not a ring.

The case for R_0^+ and Q_0^+ are similar.

Remark 2.15. $Z_0^+ = \{ x \in Z : x \geq 0\}$ is not a regular semiring.

Let us show it by an example: Suppose x = 3.

Now

$$3.y.3 = 3 \Rightarrow 3y = 1 \Rightarrow y = \frac{1}{3} \notin Z_0^+.$$

Hence $Z_0^+ = \{ x \in Z : x \geq 0\}$ is not a regular semiring.

3. Some Structures of Semirings

In this section, we discuss about additively inverse semiring, idempotent semiring and Boolean semiring.

Definition 3.1. Let (S; +, .) be a commutative semiring with zero (0) and identity (1). Then (S; +, .) is called ***idempotent semiring*** iff $\forall x \in S$;

$$x + x = x = xx.$$

Example 3.1(a). Let $S \subseteq [0,1]$ be such that $0, 1 \in S$. + and . are defined by

$$x + y = \max\{x , y\} \text{ and } x . y = \min \{x , y\}; \forall x, y \in S.$$

Then (S; +, .) is an idempotent semiring having 0 and 1 as its zero and identity.

Example 3.1(b). Let $X \neq \phi$ and P(X) is the power set of X. + and . are defined by $A + B = A \cup B$ and A.B $= A \cap B; \forall A, B \in P(X)$. Then (P(X); +, .) is an idempotent semiring, where ϕ and X are zero and identity of P(X).

Proposition 3.2. Let (S; +, .) be a Boolean semiring. Then

(i) $\forall x \in S, 2x = 4x$

(ii) If $x \in S$ is an additively invertible element of S, then 2x = 0.

(iii) If S has an identity 1, then 1 is the only multiplicatively invertible element of S.

Proof: (i) $\forall x \in S$,

$$\begin{aligned} 2x &= x + x \\ &= (x + x)^2 \\ &= (x+x)(x+x) \\ &= x.x+x.x+x.x+x.x \end{aligned}$$

= x+x+x+x [by Definition 3.1]

= 4x

(ii) Since $x \in S$ is an additively invertible element in S, so $\exists y \in S$ such that x + y = 0.

Then

2x+2y = 0

Now

2x = 2x +0

= 2x + 2x+2y

= 4x +2y

= 2x +2y [by (i)]

= 0

(iii) Since S has an identity 1, then $\forall x \in S$,

1.x = x = x.1

Suppose $y \in S$ is inverse of x. Then xy =1.

Now

x = x.1

= x.(xy)

= (x.x)y

= xy [by Definition 3.1]

=1

This shows that 1 is the only multiplicatively invertible element of S.

Proposition 3.3. Let (S; +,.) be an additively inverse semiring and $x \in S$. $x' \in S$ is unique inverse of x. Then $x + x'$ and $x' + x$ are additive idempotents.

Proof: Here $x + x' + x = x, \quad x' + x + x' = x'$

So

$$(x + x') + (x + x') = x + (x' + x + x')$$
$$= x + x'.$$

Again

$$(x' + x) + (x' + x) = x' + (x + x' + x)$$
$$= x' + x.$$

Proposition 3.4. In an additively inverse semiring S, if e, f are additive idempotents then so are $e + f$ and $f + e$.

Proof: Let $(e + f)' = g$, where g is unique.

Then

$$(e + f) + g + (e + f) = e + f$$

and

$$g + (e + f) + g = g.$$

Let $h = g + e$.

Then $(e+f)+h+(e+f)=(e+f)+(g+e)+(e+f)$

$$= e+f+g+e+e+f$$
$$= e+f+g+e+f$$
$$= (e+f)+g+(e+f)$$
$$= (e+f)$$

Again $h+(e+f)+h=(g+e)+(e+f)+(g+e)$

$$= g+(e+e)+f+g+e$$
$$= g+e+f+g+e$$
$$= (g+e)+f+(g+e)$$
$$= (g+e)$$
$$= h$$

So $(e+f)'=h$

So $g=h=g+e.$

By similar consideration,

$$g=f+g.$$

Let $h_1=f+g$.

$$(e+f)+h_1+(e+f)=(e+f)+(f+g)+(e+f)$$
$$= e+f+f+g+e+f$$
$$= e+f+g+e+f$$
$$= (e+f)+g+(e+f)$$
$$= e+f$$

$$h_1+(e+f)++h_1=(f+g)+(e+f)+(f+g)$$
$$= f+g+e+f+f+g$$
$$= f+g+e+f+g$$
$$= f+g+(e+f)+g$$
$$= f+g$$
$$= h_1$$

So $(e+f)'=h_1$ and $g=h_1=f+g$.

Thus $g+g=(g+e)+(f+g)$

$$= g+(e+f)+g$$
$$= g$$

So g is additively idempotent.

It follows that $g+g+g=g+g=g$

and so $g'=g.$

But $g'=((e+f)')'=e+f.$

So $e+f=g$ and $e+f$ is additively idempotent.

Proposition 3.5. In an additive inverse semiring, any two additive idempotents are additively commutative.

Proof : Let e, f be additive idempotents.

Then the Proposition 3.4, $e+f$, $f+e$ are also additive idempotents.

Now $(e+f)+(f+e)+(e+f) = e+(f+f)+(e+e)+f$

$$= e+f+e+f$$
$$= (e+f)+(e+f)$$
$$= e+f$$

$$(f+e)+(e+f)+(f+e) = f+(e+e)+(f+f)+e$$
$$= f+(e+e)+(f+f)+e$$
$$= (f+e)+(f+e)$$
$$= f+e$$

So $(e+f)' = f+e$

Also $(e+f)' = e+f$

So $e+f = f+e$.

Proposition 3.6. Let (S; +,.) be an additively inverse semiring, then $\forall x, y \in S$,

(i) $(x')' = x$

(ii) $(x+y)' = y'+x'$

(iii) $(xy)' = x'y = xy'$

(iv) $x'y' = xy$.

Proof : (i) This follows from the definition of x' and its uniqueness:

$$x + x' + x = x, \quad x' + x + x' = x'$$
$$\Rightarrow (x')' = x.$$

(ii) $(x+y)+(y'+x')+(x+y) = x+(y+y')+(x'+x)+y$

$$= x+(x'+x)+(y+y')+y \quad \text{[By the Proposition 3.5]}$$
$$= (x+x'+x)+(y+y'+y)$$
$$= x+y$$

Again

$(y'+x')+(x+y)+(y'+x') = y'+(x'+x)+(y+y')+x'$

$$= x+(x'+x)+(y+y')+y$$
$$= (x+x'+x)+(y+y'+y)$$
$$= x+y$$

So $(x+y)' = y'+x'$

(i) By definition of y'

$$y + y' + y = y, \quad y' + y + y' = y'$$
$$\Rightarrow xy = x(y+y'+y), \; xy' = x(y'+y+y')$$

$$\Rightarrow xy = xy + xy' + xy \ , xy' = xy' + xy + xy'$$

$$\Rightarrow (xy)' = xy'.$$

From

$$x + x' + x = x, \quad x + x' + x = x$$

$$\Rightarrow (x + x' + x)y = xy, \ (x' + x + x')y = x'y.$$

$$\Rightarrow xy + x'y + xy = xy, \ x'y + xy + x'y = x'y.$$

$$\Rightarrow (xy)' = x'y.$$

Therefore $(xy)' = x'y = xy'$

(iv) $(xy)' = xy' \Rightarrow ((xy)')' = (xy')' \Rightarrow xy = x'y'.$

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REFERENCES

1. C. Reutenauer and H. Straubing, Inversion of matrices over a commutative semiring, *J. Algebra*, 88(1984) 350-360.
2. D.E. Rutherford, Inverses of Boolean matrices, *Proc Glasgow Math Asssoc.,* 6 (1963) 49-53.
3. I. Kaplansky, *Fields and Rings*, The University of Chicago Press, Chicago, 1969.
4. K.R. Goodearl, *Von Neumann Regular Rings*, Pitman, London, 1979.
5. Li Fang, Regularity of semigroup rings, *Semigroup Forum*, 53 (1996) 72-81.
6. M Petrich, *Introduction to Semiring*, Charles E Merrill Publishing Company, Ohio, 1973.
7. M.K.Sen and S.K. Maity, Regular additively inverse semirings, *Acta Math. Univ. Comenianae,* LXXV, 1 (2006) 137-146
8. N. Sirasuntorn, S. Sombatboriboon and N. Udomsub, Inversion of matrices over Boolean semirings, *Thai Journal of Mathematics,* 7(1) 1 (2009) 105-113.
9. P.H. Karvellas, Inversive semirings, *J. Austral. Math Soc.*, 18 (1974) 277-288.
10. R.D Luce, A note on Boolean matrix theory, *Proc Amer. Math Soc.*, 3(1952) 382-388.
11. S.Chaopraknoi, K.Savettaseranee and P. Lertwichitsilp, On regular matrix semiring, *Thai Journal of Mathematics*, 7(1) (2009) 69-75.
12. S. Sombatboriboon, W. Mora and Y. Kemprasit, Some results concerning invertible matrices over semirings, *Science Asia,* 37(2011) 130-135.
13. T. Vasanthi and M. Amala, Some special classes of semirings and ordered semirings, *Annals of Pure and Applied Mathematics*, 4(2) (2013) 145-159.
14. T. Vasanthi and N. Solochona, On the additive and multiplicative structure on semirings, *Annals of Pure and Applied Mathematics*, 3(1) (2013) 78-84.