Annals of Pure and Applied Mathematics
Vol. 5, No.2, 2014, 168-182
ISSN: 2279-087X (P), 2279-0888(online)
Published on 3 March 2014
www.researchmathsci.org

On Hausdorff and Compact U-Spaces

Nasima Akhter[1]***, Swapan Kumar Das***[2] **and *Subrata Majumdar***[3]

[1]Department of Mathematics, University of Rajshahi, Rajshahi, Bangladesh
Email: nasima.math.ru@gmail.com

[2]Department of Mathematics, University of Rajshahi, Rajshahi, Bangladesh
Email: swapanuoda@yahoo.com (corresponding author)

[3]Department of Mathematics, University of Rajshahi, Rajshahi, Bangladesh
Email: subrata41majumdar@gmail.com

Received 31 January 2014; accepted 28 February 2014

Abstract. This is the first of a series of papers, where **U-**spaces and some related concepts have been introduced and their properties have been studied. It is an extension of study of supratopological spaces. In this paper Hausdorff, normal, regular and completely regular U-spaces as well as compact and locally compact U-spaces have been defined and studied. It has been shown that Hine-Borel Therom does not hold for the usual U- space $\mathbb{R}$. One-point-compactification of U-spaces has been introduced and its properties have been studied also.

Keywords: Supratopology,U-space, Hausdorff, normal, regular, completely regular, compact, locally compact, compactification

AMS Mathematics Subject Classification (2010): 54A40, 54D10, 54G12, 54D15

1. Introduction

In this paper we have introduced and studied U- spaces and their properties. These spaces have been called supratopological spaces by some authors ([1, 2,8,12]). However the U-spaces we have considered here are more general than those considered in the above papers. In this general set up we have studied some properties of Hausdorff, normal, regular and completely regular U-spaces, and compact and locally compact U- spaces. We have also defined one- point-compactification of locally compact U- space and studied its properties.

2. Definitions and preliminaries

We begin with some basic definitions and examples related to U- spaces.

Definition 2.1. [11] A subfamily $\mathscr{M}$ of the power set P(X) of a nonempty set X is called a **minimal structure** (briefly M - structure) on X if, $\Phi \in \mathscr{M}$ and $X \in \mathscr{M}$.

By (X, $\mathcal{M}$), we denote a nonempty subset X with a minimal structure $\mathcal{M}$ on X and call it an M-space. Each member of $\mathcal{M}$ is said to be M-open and complement of an M-open set is said to be M-closed set.

Example 2.1. Let X = {a, b, c, d}, $\mathcal{M}$ = {X, Φ, {a, b}, {b, c}}. Then (X, $\mathcal{M}$) is an M - space.

Definition 2.2. A U-**structure** on a nonempty set X is a collection $\mathcal{U}$ of subsets of X having the following properties:

(i) Φ and X are in $\mathcal{U}$,

(ii) Any union of members of $\mathcal{U}$ is in $\mathcal{U}$.

The ordered pair (X, $\mathcal{U}$) is called a **U- space**. A U- space which is not a topological space is called a **proper U- space**. The members of $\mathcal{U}$ are called **U-open sets** and the complement of a U- open set is called a **U- closed set**.

A U-structure and a U-space have been called a supratopology and a supratopological space respectively by some authors (see [1,2,8,12]).

In general we have

Topological space ⇒ U-space ⇒ M-space

Topological space $\not\Leftarrow$ U-space $\not\Leftarrow$ M-space

Example 2.2. Let X = {a, b, c, d}, $\mathcal{U}$ = {X, Φ, {a, b}, {a, c}, {a, b, c}}. Here (X, $\mathcal{U}$) is a U- space but not a topological space.

Example 2.3. Let X be a totally ordered set with order relation ≤ and $\mathcal{U}$ the set of all unions of the sets of the forms {x∈ X: x < a} and {x∈ X: x > b}. Then $\mathcal{U}$ is called **order U- structure on X.**

Example 2.4. Let ℝ denote the real numbers and let $\mathcal{U}$ consist of the empty set, all open rays and their unions, then (ℝ, $\mathcal{U}$) is a U-space. This U- space will be called the **usual U-space** ℝ and will be denoted simply by ℝ. We note that $\mathcal{U}$ is not a topology on ℝ, since (2,3) = $(-\infty, 3) \cap (2, \infty) \notin \mathcal{U}$.

Definition 2.3. Let (X,$\mathcal{U}$) be a U- space and $\Phi \neq A \subseteq X$ .Let $\mathcal{U}'$ = {A ∩ G | G∈ $\mathcal{U}$ } is a U- structure in A.For, $\underset{\alpha}{\cup}(A \cap G_\alpha) = A \cap (\underset{\alpha}{\cup} G_\alpha)$ and $\underset{\alpha}{\cup} G_\alpha \in \mathcal{U}$. Then (**A, $\mathcal{U}'$) is a** U-space and is called a U-subspace of (X,$\mathcal{U}$). Also, we say that A is a U-subspace of X.

Example 2.5. Let X = (0, 1) and $\mathcal{U}$ the union of the sets{(0,b) : b∈ ℝ, 0 < b < 1} and {(a,1) : a∈ ℝ, 0 < a < 1}. Then (X, $\mathcal{U}$) is a U- space but not a topological space , since

$\left(\frac{1}{2},1\right)\cap\left(0,\frac{2}{3}\right)=\left(\frac{1}{2},\frac{2}{3}\right)\notin\mathcal{U}$. In fact this is the U- space obtained by considering (0,1) as a U- subspace of $\mathbb{R}$ with the usual U-structure .

In the usual U-space $\mathbb{R}$, every singleton set {a} is closed in $\mathbb{R}$, since $\{a\} = (-\infty, a] \cap [a, \infty)$. However, every finite set need not be closed.

Definition 2.4. A subset A of a topological space X is said to be:
1. Pre-open [5] if $A \subseteq \text{Int(Cl(A))}$
2. Semi-open [5] if $A \subseteq \text{Cl(Int(A))}$
3. α -open [10] if $A \subseteq \text{Int(Cl(Int(A)))}$
4. β-open [10] if $A \subseteq \text{Cl(Int(Cl(A)))}$
5. δ-open [13] if $\text{Int(Cl(A))} \subseteq \text{Cl(Int(A))}$.
6. b-open [1,5] if $A \subseteq \text{Cl(Int(A))} \cup \text{Int(Cl(A))}$
7. *b-open [5] if $A \subseteq \text{Cl(Int(A))} \cap \text{Int(Cl(A))}$
8. b**-open [5] if $A \subseteq \text{Int(Cl(Int(A)))} \cup \text{Cl(Int(Cl(A)))}$
9. **b- open [5] if $A \subseteq \text{Int(Cl(Int(A)))} \cap \text{Cl(Int(Cl(A)))}$
10. Locally open [6] if $A = G \cup F$, for an open subset G and a closed subset F of X.
11. Locally closed [6] if $A = G \cap F$, for some open subset G and closed subset F of X.

Remark 2.1. Let X be a topological space. Let the classes of all b-open (resp. b*-open, b**- open, **b- open) sets in X be denoted by b(X) (resp. b*(X), b**(X), **b(X)). We shall now consider which of (X, PO(X)), (X, SO(X)), (X, β(X)), (X, LO(X)), (X, LC(X)), (X, α(X)), (X, δ(X)) and (X, b(X)), (X, b*(X)), (X, b**(X)), (X, **b(X)) are **M-spaces and which are U-spaces.**

(i) (X, α(X)) is a topological space. So, it is both an M-space and U-space.
(ii) (X, PO(X)) is a U- space.
(iii) (X, SO(X)) is a U-space.([6], Them.15(i), (ii)).
(iv) (X, β(X)) is a U-space, but not a topological space. ([6], Them.18(i)).
(v) (X, LO(X)), (X, LC(X)), (X, δ(X)) are not U-spaces but are M-spaces. ([6], Them. 16(i), 17(i), 19(i)).
(vi) (X, b(X)), (X, b*(X)), (X,b**(X)) and (X, **b(X)) are **U-spaces.**

Remark 2.2. Let (X, $\mathcal{U}$) be a U- space. Let $\mathcal{T}_{\mathcal{U}}$ denote the topology generated by $\mathcal{U}$ on X. This will be called the **topology indeed by $\mathcal{U}$**. Also, if (X, $\mathcal{T}$) is a topological space, (X, $\mathcal{T}$) is a U- space. Also, for any subcollection or supercollection $\mathcal{U}$ of $\mathcal{T}$ in $\mathcal{P}$(X) which is closed under union is a U- structure on X. (X, $\mathcal{U}$) is supratopology on X, associated with $\mathcal{T}$. A. S. Mashhour and others have considered and studied these supratopologies associated with a topology. We have dealt with U- spaces in general.

Definition 2.5. Let (X, $\mathcal{U}$) be a U-space. For a subset A of X, the **U-closure of A** (${}_U$**Cl(A)**) and the **U-interior of A** (${}_U$**Int(A)**) are defined as follows:

$$_U\mathbf{Cl(A)} = \cap\{F{:}A \subseteq F,\ F^c \in \mathcal{U}\},\ _U\mathbf{Int(A)} = \cup\{U{:}U \subseteq A,\ U \in \mathcal{U}\}.$$

Clearly**,** we have $_U$**Cl(A)** is U- closed and $_U$**Int(A)** is U-open.

Lemma 2.1. Let X be a U-space and A a subset of X. Then $x \in {}_U\mathrm{Cl}(A)$ if and only if $G \cap A \neq \Phi$, for every U-open set G containing x.
The proof is exactly similar to that in the case of topological spaces.

As in the case of supratopological spaces [8], we define 3 types of continuity in the following.

Definition 2.6. Let **(X, $\mathcal{U}$)** and **(Y, $\mathcal{U}'$)** be two U- spaces. A function f: X $\rightarrow$ Y is said to be **U-continuous** if for each U-open set G′ in Y, $f^{-1}(G')$ is U-open set in X.

Example 2.6. Let X = {a, b, c, d }, $\mathcal{U}$ = {X,Φ,{a},{a, b},{a, c, d},{b, c, d}}
Y = {p, q, r}, $\mathcal{U}'$ = {Y,Φ,{p},{p, q},{p, r},{q, r}}.Let f: X $\rightarrow$ Y be defined by f(a) = p, f(b) = q, f(c) = r, f(d) = r. Then f is U- continuous.
Here **(X, $\mathcal{U}$)** and **(Y, $\mathcal{U}'$)** are two U- spaces but not a topological spaces.

Definition 2.7. Let **(X, $\mathcal{U}$)** be a U- space and **(Y, $\mathcal{T}$)** a topological space. A function f: X $\rightarrow$ Y is said to be $\overline{U}$ **- continuous** if for each open set H in Y, $f^{-1}(H)$ is U-open set in X.

Example 2.7. Let X= {a, b, c}, $\mathcal{U}$ = {X,Φ,{a},{b, c},{a, c}}. **Y =** {p, q, r },
$\mathcal{T}$ = {Y,Φ,{p},{p, q},{p, r}}. **(X, $\mathcal{U}$)** is a U- space but not a topological space and (Y, $\mathcal{T}$) is a topological space. The function f: X $\rightarrow$ Y defined by f(a) = r,f(b) = q, f(c) =q, f is $\overline{U}$ - continuous.

Definition 2.8. Let **(X, $\mathcal{T}$)** be a topological space and **(Y, $\mathcal{U}$)** a U- space. A function f: X $\rightarrow$ Y is said to be U^* **-continuous** if for each U-open set H in Y, $f^{-1}(H)$ is open set in X.

Example 2.8. Let X = {a, b, c, d}, $\mathcal{T}$ = {X, Φ, {a},{b},{c},{a, b},{b, c},{a, c}, {c, d},{a, b, c}, {a, c, d},{b, c, d}}. Then **(X, $\mathcal{T}$)** is a topological space.**Y = {**p, q, r }**,** $\mathcal{U}$ = {Y,Φ,{p},{p, q},{p, r}, {q, r}}. Then **(Y, $\mathcal{U}$)** is a U- space but not a topological space. The function f: X $\rightarrow$ Y defined by f(a) = p, f(b) = q, f(c) = r, f(d) = r, f is U^*- continuous.

3. Compact U-spaces

Definition 3.1. Let **(X, $\mathcal{U}$)** be a U- space. A **U- open cover** of a subset K of X is a collection $\{G_\alpha\}$ of U - open sets such that $K \subseteq \bigcup_\alpha G_\alpha$.

Definition 3.2. A U-space X is said to be c**ompact** if for every U-open cover of X has a finite sub-cover.

A subset K of a U- space X is said to be c**ompact** if every U-open cover of K has finite sub- cover.

Example 3.1. Let X = $\mathbb{N}$, $\mathcal{U}$ = {**2** $\mathbb{N}$**, 4** $\mathbb{N}$**, 8** $\mathbb{N}$**, 16**$\mathbb{N}$**, ……. , 2**^{n} $\mathbb{N}$**, ...,** $\mathbb{N}$, Φ}. Then X is a compact U- space.

Let $\Phi \neq$ A $\subseteq$ X and $\mathfrak{C}$ be a U open cover of A. Let n_0 be smallest +ve integer such that $\mathbf{2}^{n_0}\,\mathbb{N} \in \mathfrak{C}$**.** Then $A \subseteq \mathbf{2}^{n_0}\,\mathbb{N}$**.** So {$\mathbf{2}^{n_0}\,\mathbb{N}$ } is a finite sub-cover of $\mathfrak{C}$**.** Therefore every subset of X is compact.

Example 3.2. Let X = $\mathbb{N}$ and $\mathcal{U}$ = {m $\mathbb{N}$**:** m$\in$ $\mathbb{N}$ **}**$\cup$**{**Φ}. Then X is a compact U- space.

Heine-Borel Theorem is an important result for compactness in Topology. This states that a subspace A of the real line $\mathbb{R}$ is compact if and only if A is closed and bounded.

However, the corresponding theorem does not hold for the usual U- space $\mathbb{R}$. For, $\mathbb{N}$ is a compact subspace of the usual U- space $\mathbb{R}$ but it is neither closed nor bounded.

As for topological spaces, the following result is true.

Theorem 3.1. Let **(X,** $\mathcal{U}$) and **(Y,** $\mathcal{U}'$) be two U- spaces. If f: X $\to$Y is a U-continuous function and B is a compact subset of U- space X, then f(B) is compact.

Proof: Let {H_i: $i \in I$ } be any U-open cover of f(B). For each x$\in$ B, there exists i(x) $\in$ I such that f(x) $\in H_{i(x)}$. Since f is U- continuous, there exists a U-open set G(x) containing x such that f(G(x))$\subseteq H_{i(x)}$. The family {G(x): x $\in$ B} is a U- open cover of B. Since B is compact, there exists a finite number of points, say $x_1,x_2,x_3,\ldots\ldots,x_n$ in B such that $B \subseteq \cup\{G(x_k): x_k \in B, 1 \le k \le n\}$. Therefore, we have

$$f(B) \subseteq \cup\{f(G(x_k)): x_k \in B, 1 \le k \le n\} \subseteq \cup\{H_{i(x_k)}: x_k \in B, 1 \le k \le n\}.$$

Thus f(B) is compact.

We can similarly prove the following two results:

Theorem 3.2. Let **(X,** $\mathcal{U}$) be a U- space and **(Y,**$\mathcal{T}$) a topological space. If f: X $\to$Y is a $\overline{U}$ -continuous function and B is a compact subset of U- space X, then f(B) is compact.

Theorem 3.3. Let **(X,** $\mathcal{T}$ **)** be a topological space and **(Y,** $\mathcal{U}$) be a U- space. If f: X $\to$Y is a U^*-continuous function and B is a compact subset of U- space X, then f(B) is compact U- space.

Theorem 3.4. Every U- closed subspace of a compact U- space is compact.

Proof: Let X be a compact U-space and F be closed subspace of X. Let $\{V_i\}$ be U-open cover of F. So $F \subseteq \cup V_i$ and $V_i = G_i \cap F$, where G_i is a U-open set of X. Therefore $F^C \cup \{G_i\}$ is a U-open cover of X. Since X is a compact U- space. There exists i_1, i_2, i_3,........,i_n such that $X = F^C \cup G_{i_1} \cup G_{i_2} \cup \ldots\ldots\cup G_{i_n}$

$\therefore F \subseteq V_{i_1} \cup V_{i_2} \cup \ldots\ldots\cup V_{i_n}$. $\therefore$ F is compact.

Definition 3.3. A U-space X is called **Hausdorff** if, for each x, y $\in$ X, $x \neq y$, there exists disjoint U-open sets G and H in X such that $x \in G$, $y \in H$.

Example 3.3. Let X = {a, b, c, d},
$\mathcal{U}$ ={{a},{d},{b, c},{b, d}, {a, d}, {a, c},{a, b, c},{b, c, d},{a, c, d},{a, b, d}, X, Φ }.
Then (**X**, $\mathcal{U}$) is a Hausdorff U- space.

Example 3.4. The usual U- space $\mathbb{R}$ is Hausdorff , for any x, y $\in \mathbb{R}$ with $x \neq y$ (say x < y), there exist two disjoint U- open sets $(-\infty, \frac{x+y}{2})$ and $(\frac{x+y}{2}, \infty)$ containing x and y respectively.

Example of a U- space which is not Hausdorff is given below.

Example 3.5. Let X be an infinite set and $\mathcal{U}$ = { X, Φ,{$G \subseteq X \mid G^c$ is a singleton set}} **.**
Then (**X**, $\mathcal{U}$) is a proper U- space which is not Hausdorff.

Theorem 3.5. Every subspace of Hausdorff U- space is Hausdorff.
Proof: It is trivial.

Theorem 3.6. In a topological space every compact subspace of a Hausdorff space is closed.

However, we note that the following is true.
Remark 3.1. A compact subset of a Hausdorff U- space need not be closed. Its truth is proved by the following example:

Example 3.6. Let A = $\{1,2,3\} \subseteq \mathbb{R}$**,** then clearly A is compact U- space, but it is not closed. Because every U-closed set in $\mathbb{R}$ is of the form [b, ∞), or ($-\infty$, a] or their intersection.

4. Separation axioms and Compactification in U- spaces

Definition 4.1. A U- space X is a **T_0 -U-space** if for each x, y$\in$ X, with $x \neq y$, there exist two distinct U-open sets G and H in X such that $x \in G$, $y \in H$.

Example 4.1. Let X = {a, b, c, d}, $\mathcal{U}$ = {{a},{d},{b, c}, {b, d},{a, d},{a, c},{a, b, c}, {a, b, d}, {b, c, d}, {a, c, d}, X, Φ }.
Then (X, $\mathcal{U}$) is a T_0- U space. But (X, $\mathcal{U}$) is not a topological space.

Definition 4.2. A U- space X is **T_1-U-space** if for each x, y$\in$ X, x $\neq$ y, there exist two U-open sets G and H in X such that x$\in$ G, y$\notin$ G and x$\notin$ H, y$\in$ H.

Example 4.2. Let X be an infinite set. Let $\mathcal{U}$ consist of the sets $\{a\}^c$, for each a$\in$ X and their unions. Clearly, X, $\Phi\in$ $\mathcal{U}$. Then (X, $\mathcal{U}$) is a T_1- U space.However, (X,$\mathcal{U}$) is not a topological space. Since $\{a\}^c \cap \{b\}^c = \{a, b\}^c \notin \mathcal{U}$.

Example 4.3. Let X = {a, b, c}, $\mathcal{U}$ = {{a, b},{a, c}, X, Φ }.
Then (X, $\mathcal{U}$) is T_0-U-space but not T_1-U-space.
Here T_1-U-space $\Rightarrow$ T_0-U- space, but T_0-U- space $\not\Rightarrow$ T_1-U- space.

Theorem 4.1. A U- space X is T_1-U-space iff every subset of X which consisting of exactly one point of X is U- closed.
Proof: Let X be a T_1-U-space and x$\in$ X. We shall show that X – {x} is U-open. Let y$\in$ X – {x}. Since X is T_1-U-space, for each y$\in$ X, y$\neq$x, there exist U-open set G_y such that y$\in$ G_y but x $\notin$ G_y. So, $G_y \subseteq$ X – {x}. Therefore X – {x} is U- open.

Conversely, let every subset containing one point of X be U-closed and let x, y$\in$ X and x $\neq$ y . Since {x} and {y} are U- closed, G = X – {y}, H = X – {x} are U- open and x$\in$ G, y$\notin$ G and x$\notin$ H, y$\in$ H. Therefore X is T_1-U-space.

Definition 4.3. A Hausdorff U- space is called a T_2-U-space.

Example 4. 4. Let X = {a, b, c}, $\mathcal{U}$ = {X, Φ, {a}, {b},{b, c},{a, c}, {a, b}}. Then (X,$\mathcal{U}$) is a U- space but not a topological space. Here (X,$\mathcal{U}$) is a **T_2- U- space**.
(X, $\mathcal{U}$) in Ex.- 4.2 is a T_1-U-space but it is not a T_2- U- space.
Hence every T_2 – U- space is a T_1-U-space, but not conversely.

Definition 4.4. Let (X, $\mathcal{U}_X$) and (Y, $\mathcal{U}_Y$) be U- spaces. (X$\times$Y, $\mathcal{U}$), where $\mathcal{U}$ is a collection of subsets of X$\times$Y, is called **the product of X with Y** if $\mathcal{U}$ is the U- structure on X$\times$Y generated by $\left(\bigcup_{x\in X}\{\pi_x^{-1}G_x\}\right)\cup\left(\bigcup_{y}\{\pi_y^{-1}G_y\}\right)$, where π_x: X$\times$Y$\rightarrow$X and π_y: X$\times$Y$\rightarrow$Y are the projection maps.

Hence if (X$\times$Y, $\mathcal{U}$) is the product of (X, $\mathcal{U}_X$) with (Y, $\mathcal{U}_Y$), then $\mathcal{U}$ is the smallest U - structure on X$\times$Y such that the projection maps π_x: X$\times$Y$\rightarrow$X and π_y : X$\times$Y$\rightarrow$Y are U-continuous.

In general, let $\{X_\alpha, \mathcal{U}_\alpha\}$be any non- empty family of non- empty U- spaces. Then, ($\prod_\alpha X_\alpha$, $\mathcal{U}$), where $\mathcal{U}$ is a collection of subsets of $\prod_\alpha X_\alpha$, is called **the product of $\{X_\alpha, \mathcal{U}_\alpha\}$** if $\mathcal{U}$ is the U- structure on $\prod_\alpha X_\alpha$ generated by $\bigcup_\alpha \{\pi_\alpha^{-1}(U_\alpha) | U_\alpha \in \mathcal{U}_\alpha\}$, where $\pi_\alpha : \prod_\alpha X_\alpha \to X_\alpha$ is the projection map.

Theorem 4.2. (X$\times$Y, $\mathcal{U}$) is the product of (X, $\mathcal{U}_1$) with (Y, $\mathcal{U}_2$) iff $\mathcal{U}$ is the U-structure generated by $\{G_1\times Y: G_1 \in \mathcal{U}_1\} \cup \{X\times G_2: G_2 \in \mathcal{U}_2\}$.

Our next theorems are generalizations of (Theorems- 2.2- 2.4, p. 102 -103, in [7]).

Theorem 4.3. The product of any nonempty class of Hausdorff U- spaces is Hausdorff.
Proof: Let $\{X_i\}$ be the product of a nonempty class of Hausdorff U- spaces X_i and X $=\prod X_i$. Suppose x, y$\in$ X, x $\neq$ y. If x = $\{x_i\}$ and y = $\{y_i\}$ are two distinct points in X, then we must have $x_{i_0} \neq y_{i_0}$ for at least one index i_0. Since X_{i_0} is a Hausdorff U- space, there exist two disjoint U- open sets U and V of X_{i_0} such that $x_{i_0} \in$ U and $y_{i_0} \in$ V. Let G = $\prod_i G_i$ and H $=\prod_i H_i$, where U = G_{i_0} and V = H_{i_0} and for $i \neq i_0$, $G_i \cup H_i = X_i$. Thus G and H are two disjoint U- open sets of X and x $\in$ G and y$\in$ H. Therefore X is Hausdorff.

Definition 4.5. Let (X, $\mathcal{U}$) be a U- space and R an equivalance relation on X. For each U$\in$ $\mathcal{U}$, let $U^{/} = \{\text{cls } x \mid x\in \mathcal{U}\}$. Let $\mathcal{U}^{/} = \{U^{/} \mid U\in \mathcal{U}\}$. Then $\mathcal{U}^{/}$ is a U- structure on $\frac{X}{R}$. ($\frac{X}{R}, \mathcal{U}^{/}$) will be called the **usual U-space** $\frac{X}{R}$**,** unless otherwise stated, $\frac{X}{R}$ will denote this U- space.

Theorem 4.4. Let X be a U- space and R is an equivalence relation on X. If R is a U-closed subset of the product U- space X$\times$X, then $\frac{X}{R}$ is Hausdorff .

Proof: Let p : X$\to \frac{X}{R}$ be projection mapping i. e. p(x) = clsx. Let z, z$'\in \frac{X}{R}$.
So z = p(x), z$'$ = p(x$'$), where x, x$'\in$ X. Since R is a U- closed subset of X$\times$X, there exist two U-open sets U and V such that (x, x$'$) $\in$ U$\times$V $\subseteq$R$'$. Since p is U- open mapping, p(U), p(V) are U-open. Clearly, z $\in$ p(U), z$'\in$ p(V). Since U$\times$V $\subseteq$R$'$, p(U)$\cap$p(V) = Φ. Hence $\frac{X}{R}$ is Hausdorff.

Theorem 4.5. Let X be a U- space and Y a Hausdorff U- space and let $f : X \rightarrow Y$ be a U- continuous mapping. Then $\frac{X}{R(f)}$ is Hausdorff.

[Here R(f) is an equivalence relation of X,given by $(x, x') \in R(f) \Rightarrow f(x) = f(x')$].

Proof: Let clsx and clsy be two distinct elements of $\frac{X}{R(f)}$. So f(x) and f(y) are two distinct elements of Y. Since Y is Hausdorff , there exist two disjoint U-open sets G and H of Y such that $f(x) \in G$ and $f(y) \in H$. Since f is U- continuous, $f^{-1}(G)$ and $f^{-1}(H)$ are disjoint U- open sets of X. Hence $x \in f^{-1}(G)$ and $y \in f^{-1}(H)$.

Again $p : X \rightarrow \frac{X}{R(f)}$ is projection mapping, this implies that $p(f^{-1}(G))$ and $p(f^{-1}(H))$ are two disjoint U- open sets of $\frac{X}{R(f)}$ containing clsx and clsy respectively. Hence $\frac{X}{R(f)}$ is Hausdorff .

Definition 4.6. A U- space X is said to be **$U - T_{2\frac{1}{2}}$ space or, completely Hausdorff** if, for each $x, y \in X$, $x \neq y$, there exist U-open sets G and H such that $x \in G$ and $y \in H$ and $\overline{G} \cap \overline{H} = \Phi$.

Example 4.5. Let X = {a, b, c, d}, $\mathcal{U}$ = {X, Φ, {a, b},{a, c},{a, d},{ b, c}, {b, d}, {c, d}, {a, b, c},{a, b, d},{a, c, d},{b, c, d}}. Then X is a **proper completely Hausdorff U-space.**

Definition 4.7. A U- space X is called **regular** if for any U- closed set F of X and any point $x \in X$, such that $x \notin F$ there exist two disjoint U-open sets G and H such that $x \in G$ and $F \subseteq H$.

For U- spaces, ‘Hausdorff’ and 'regular' are independent concepts.

Example 4.6. (A proper U- space which is regular but not Hausdorff).

Let X = {a, b, c, d}, $\mathcal{U}$ = {X, Φ, {a},{d},{a, d},{a, b, c},{b, c, d}}. Here the U-closed sets are X, Φ, {a},{d},{b, c},{a, b, c},{b, c, d}. We easily see that X is a regular but it is not Hausdorff , since b and c cannot be separated by disjoint U- open sets. Also (X, $\mathcal{U}$) is not a topological space.

Example 4.7. (A proper U-space which is Hausdorff but not regular).

Let X = $\mathbb{R}$ and $\mathcal{U}$ is the structure generated by $\mathcal{U}_1 \cup \mathcal{U}_2$, where $\mathcal{U}_1$ is the usual space on $\mathbb{R}$ and $\mathcal{U}_2 = \{\mathbb{Q}^c\}$, where Q is the set of all rational numbers. Then (X, $\mathcal{U}$) is a proper Hausdorff U- space, since $\mathcal{U}_1 \subseteq \mathcal{U}$.

If F = $\mathbb{Q}$ and x is an irrational number, then F is U-closed, since $\mathbb{Q}^c \in \mathcal{U}_2$ and $x \notin F$. But x and F cannot be separated by disjoint U- open sets.

Here (X, $\mathcal{U}$) is not regular.
Thus a Hausdorff U- space need not be regular.

Definition 4.8. A U- space X is said to be **completely regular** if for any U- closed subset F of X and x$\in$X which does not belongs to F, there exists a U- continuous function f: X $\to$ [0, 1] such that f(x) = 0 and f(F) = 1. Here [0, 1] is considered as a subspace of the usual U- space $\mathbb{R}$.

Example 4.8. Let X = [0,1] and $\mathcal{U}$ = {X, Φ,{{[(a, 1)], [(0, b)] $\mid$ 0$\leq$a, b$\leq$1}and their unions}}. Then the U – open sets of X are X, Φ, and the sets of the form [(0,b)],[(a,1)] and [(0,b)]$\cup$[(a,1)], b < a.

Hence, the U- closed sets of X are of the form X, Φ, [(0, a)], [(b, 1)] and [(a, b)], a < b. [Here [(a, b)] stands for any of (a, b), (a, b], [a, b) and [a, b]. **]**

Clearly, (X, $\mathcal{U}$) is a proper U- space.
Let F be a proper U- closed set, i.e., Φ$\neq$ F$\neq$ X. Let c$\in$X, c$\notin$F.
Then, (i) F = [(a, b)], for some 0$\leq$a, b$\leq$1, a < b; or,
(ii) F = [(0, b)], or, (iii) F = [(a, 1)], 0$\leq$a, b$\leq$1.

We now consider Y = [0,1] as a subspace of the usual U- space $\mathbb{R}$. We first consider case (i) Define f: X$\to$Y by
(α) f(x) = 1, x$\in$(c, 1],
= 0, x$\in$[0, c], if c is on the left of F;
(β) f(x) = 1, x$\in$[c, 1),
= 0, x$\in$(c, 1], if c is on the right of F.
Then in both the cases of (α) and (β), f is U- continuous and f(F) = 1, f(c) = 0. Next, we consider the case (ii)
Define f: X$\to$Y by
f(x) = 1, x$\in$[c, 1],
= 0, x$\in$(0, c);
Then f is U – continuous and f(F) = 1, f(c) = 0.
Finally, we consider the case (iii)
Define f: X$\to$Y by f(x) = 1, x$\in$[0, c],
= 0, x$\in$(c, 1].
Here again f is U – continuous and f(F) = 1, f(c) = 0.
Hence (X, $\mathcal{U}$) is completely regular.

Comment: The above U- space X of Example 4.8 is also Hausdorff, normal and regular. We prove these below:

(i) Let x, y$\in$X, x$\neq$y. Then for the disjoint U- open sets G_1 = [0, $\frac{x+y}{2}$) and G_2 = ($\frac{x+y}{2}$, 1], x$\in G_1$, y$\in G_2$. **Thus, X is Hausdorff**.

(ii) Let F_1 and F_2 be two disjoint U- closed sets in X. We shall show that there are disjoint U- open sets G_1, and G_2 such that $F_1 \subseteq G_1$, $F_2 \subseteq G_2$. We see that F_1 is the form [0, a)], or [(b, 1], or [(a, b)].

If F_1= [0, a)], F_2 is the form (a, 1], or [(c, 1], or [(c, d)], for some c > a. In the first two cases, both F_1 and F_2 are U- open sets also, we take $G_1 = F_1$, $G_2 = F_2$.

If F_2 = [(c, d)], we take $G_1 = F_1$, $G_2 = (\frac{a+c}{2}, 1]$.

Here X is normal.

(iii) Similarly, we can prove that **X is regular**.

Definition 4.9. A regular U- space X is called **T_3-U- space** if for each one point subset of X is U-closed.

Definition 4.10. A T_1-U- space X is said to be **$T_{3\frac{1}{2}}$ -U- space** if X is completely regular.

Theorem 4.6. Every completely regular U- space is regular.

Proof: Let X be completely regular U- space. F is a U- closed set of X and $x \in X$ which does not belongs to F, there exists a U- continuous function f: X $\rightarrow$ [0, 1] such that f(x) = 0 and f(F) = 1.

Let a, b $\in$ [0, 1] and a < b. Then [0, a] and [b, 1] are two disjoint U- open set of [0, 1].

Therefore, $x \in f^{-1}[0, 1]$ and $F \subseteq f^{-1}[b, 1]$.

Therefore X is regular.

One can prove that a subspace of regular (a completely regular) U- space and a product of regular (a completely regular) U- spaces is regular (completely regular).

Definition 4.11. A U- space X is said to be n**ormal** if for each pair disjoint U- closed sets F_1 and F_2 , there exist U- open sets G_1 and G_2 such that $F_1 \subseteq G_1$, $F_2 \subseteq G_2$ and $G_1 \cap G_2 = \Phi$.

Theorems in U- spaces corresponding to the standard theorems regarding regular, normal and completely regular topological spaces can be shown to be valid. In particular, Urysohn's Lemma and Tietze Extension Theorem have their analogues for U- spaces.

We shall give here examples to show that proper regular and normal U- spaces exist and are distinct.

Example 4.9 (A proper U- space which is normal and regular.)

Let X = {a, b, c, d}, $\mathcal{U}$ = {X, Φ, {a}, {d}, {a, d}, {a, b, c},{b, c, d}}.

(X, $\mathcal{U}$) is a proper U- space , since {a, b, c} $\cap$ {b, c, d} = {b, c} $\notin \mathcal{U}$.

U-closed sets are X, Φ, {a}, {d}, {b, c}, {b, c, d}, {a, b, c}.

Here $\{b, c\} \subseteq \{a, b, c\}$ and $\{d\} \subseteq \{d\}$. $\{b, c\}$ and $\{d\}$ are U- closed and disjoint and there exist disjoint U- open sets containing $\{b, c\}$ and $\{d\}$ respectively. Similarly, we can show that for any pair of disjoint closed sets, there exist disjoint U- open sets containing them respectively.Hence X is U- normal space.

Here $\{b, c, d\}$ is closed set, $a \notin \{b, c, d\}$ and there exist disjoint U- open sets containing a and $\{b, c, d\}$ respectively. So, X is regular U-space.

We note that the U- space X in the above example is regular but not a T_3- U - space

Example 4.10. (A proper U- space which is normal but not regular)

Let $X = \{a, b, c, d\}$, $\mathcal{U} = \{X, \Phi, \{a, b\}, \{a, c\}, \{a, b, c\}\}$. $(X, \mathcal{U})$ is proper U- space , since $\{a, b\} \cap \{a, c\} = \{a\} \notin \mathcal{U}$.

U-closed sets are X, Φ, $\{a\}$, $\{c, d\}$, $\{b, d\}$, $\{d\}$.

Here $b \notin \{c, d\}$, $a \notin \{c, d\}$ but none of these can be separated by disjoint U- open sets. Hence $(X, \mathcal{U})$ is not regular.

However, $(X, \mathcal{U})$ is normal, since there are no pair of disjoint U- closed sets.

We shall now prove a few theorems.

Theorem 4.7. Every infinite Hausdorff U- space has countable infinite discrete U-subspaces.

Proof: Let X be an infinite Hausdorff U- space. Let x_1 and x_2 be distinct two points of X. Then there exist two disjoint U-open sets G_1 and G_2 of X such that $x_1 \in G_1$ and $x_2 \in G_2$.

Let $x_3 \in X$ which is separate from x_1 and x_2. Then there exist U-open sets H_1, H_2, H_3 and H_4 such that $x_1 \in H_1$, $x_2 \in H_2$, $x_3 \in H_3$ and $x_3 \in H_4$ and $H_1 \cap H_3 = \Phi$. Let $H_2 \cap H_4 = \Phi$. Suppose $H_3 \cap H_4 = U_3$, $H_1 = U_1$ and $H_2 = U_2$. Then U_1, U_2 and U_3 are disjoint U- open sets. Since X is an infinite, by using induction principle, we have for every $n \geq 1$, x_1, x_2, x_3, ------, $x_n \in X$ and U_1, U_2, U_3, -------, U_n are U-open sets such that for each $x_i \in U_i$ and for $i \neq j$, $x_i \neq x_j$ and $U_i \cap U_j = \Phi$, (i, j = 1, 2, 3------------,n).

Let $Y = \{ x_1, x_2, x_3, \text{-------} \}$. Then Y is a countable infinite U-subspace whose U-open sets are $\{x_i\} = Y \cap U_i$.

Definition 4.12. Let X be a U- space and let $\{x_n\}$ be a sequence in X. An element $x \in X$ is called a **limit of $\{x_n\}$** if, for each U- open set G of X with $x \in G$, then there exists a positive integer n_0 such that for each positive integer $n > n_0$, $x_n \in G$.

Theorem 4.8. The limit of every convergent sequence of a Hausdorff U- space is unique.

Proof: Let X be a Hausdorff U- space and $\{x_n\}$ be a convergent sequence of X. Assume that $x_n \to x$, $x_n \to y$ and $x \neq y$. Since X is Hausdorff U- space, there exist two disjoint U-open sets G and H of X such that. $x \in G$ and $y \in H$. Since x_1 and x_2 are limits of $\{x_n\}$, there exist two natural numbers n_1, n_2 such that $n > \max\{n_1, n_2\}$, then $x_n \in G$ and $x_n \in H$. Therefore $G \cap H \neq \Phi$ which is a contradiction.

Theorem 4.9. Let (X$\times$Y, $\mathcal{U}$) be the U- product of (X, $\mathcal{U}_1$) with (Y,$\mathcal{U}_2$). Then X$\times$Y is compact if X and Y are compact.

Proof: Let C = $\{G_\alpha\}_{\alpha\in A}$ be a U- cover of X$\times$Y. Then for each α,

G$_\alpha$ = $\bigcup_{i\in I}(G_{1,\alpha_i}\times Y)\cup\bigcup_{j\in J}(X\times G_{2,\alpha_j})$ for some G_{1,α_i} 's in $\mathcal{U}_1$ and G_{2,α_j} 's in $\mathcal{U}_2$.

Therefore,

X$\times$Y=

$$\bigcup_{\alpha\in A}[\bigcup_{i\in I}(G_{1,\alpha_i}\times Y)]\cup[\bigcup_{j\in J}(X\times G_{2,\alpha_j})]=[\bigcup_{\alpha\in A}\bigcup_{i\in I}(G_{1,\alpha_i}\times Y)]\cup[\bigcup_{\alpha\in A}\bigcup_{j\in J}(X\times G_{2,\alpha_j})]$$

Then C_1 = $\{G_{1,\alpha_i}\}_{\alpha\in A,i\in I}$ is a U_1- cover of X and C_2 = $\{G_{2,\alpha_j}\}_{\alpha\in A,j\in J}$ is a U_2- cover of Y. Since X and Y are compact, C_1 and C_2 have some finite U- sub covers, say $\{G_{1,\alpha_r,i_s}\}_{1\le r\le u,\,1\le s\le v}$ and $\{G_{1,\alpha_{r'},i_s'}\}_{1\le r'\le u',\,1\le s'\le v'}$ then $\{G_{1,\alpha_r}\times G_{2,\alpha_{r'}}\}_{1\le r\le u,1\le r'\le u'}$ is a finite sub cover of C. Therefore, X$\times$Y is compact.

Definition 4.13. A U- space X is said to be l**ocally compact** if for each x$\in$ X there exists a U- open set G containing x of X whose closure is compact.

Example 4.11. The U- space $\mathbb{R}$ is locally compact. Because, for a neighborhood of any real number x of the form S_a(x) = (-∞, x + a), a > o, $\overline{S_a}$ (x) = (-∞, x + a], which is compact. However, $\mathbb{R}$ is not a compact U- space, since the U-open cover {(-∞, a) | a$\in$ $\mathbb{R}$ } of $\mathbb{R}$ does not have a finite sub cover.

Every compact U- space is locally compact but locally compact U- space need not be compact.

Theorem 4.10. Every locally compact Hausdorff U- space is regular.

Proof: Let X be a locally compact Hausdorff U- space. Then X has one- point -compactification X_∞ and it is Hausdorff and compact U- space.

Since every compact Hausdorff U- space is regular, X_∞ is regular U- space.

Since the U- subspace of regular U- space is regular.Therefore X is regular U- space as X_∞ is U- subspace of X.

Theorem 4.11. Every locally compact Hausdorff U- space is completely regular.

Proof: Let X be a locally compact Hausdorff U- space. Then X has one -point-compactification X_∞ and it is Hausdorff and compact U- space. By Theorem 4.11, X is normal.

Let F be U- closed subset of X and x$\notin$ F. Then the definition of U- open set of X_∞,F is U- closed subset of X_∞,there exists a U- continuous function f : $X_\infty \to$[0, 1], where f(x) = 0, f(F) =1. Let the function $\overline{f}$: X$\to$[0, 1] is defined by $\overline{f}$ $\overline{f}$ (x) = f(x),

$x\in X$. Then $\overline{f}$ is U-continuous and $\overline{f}$ (x) = 0, $\overline{f}$ (F) = 1. Therefore X is completely regular.

Definition 4.14. If Y is a compact Hausdorff U- space and X is a proper U- subspace of Y whose closure equals to Y, then Y is said to be a **compactification** of U- space X.

Two compactifications Y_1 and Y_2 of U- space X are said to be **equivalent** if there is a U-homeomorphism h: $Y_1 \to Y_2$ such that h(x)= x for every $x\in X$.
If Y- X consists of a single point, then Y is called a **one-point-compactification** of X.

Theorem 4.12. A U -space X has a one- point -compactification if and only if X is locally compact but not itself compact.
Proof: To see this, let X be a locally compact U- space but not itself compact, and let Y = {y}, where $y\notin X$. Let $Z = X \cup Y$. Declare a subset V to be U- open in Z if either V is U-open in X or V is the K^c the complement of a compact U- space K in X. Then Z becomes a compact U- space, and is the one - point- compactification of X. Z will be denoted by X_∞ (as in topology) and y denoted by ∞.

Example 4.12. The one-point- compactification of the usual U-space $\mathbb{R}$ is homeomorphic with the circle. The **one -point- compactification** of $\mathbb{R}^2$ is homeomorphic to the sphere S^1.

Let S^1 denote the unit circle $\{(x, y) \in \mathbb{R}^2: x^2 + y^2 = 1\}$ regarded as a U- subspace of the product $\mathbb{R} \times \mathbb{R}$ of the usual U- space $\mathbb{R}$ with itself. The imbedding h: (0, 1) $\to S^1$ given by h(t) = $(\cos 2\pi t) \times (\sin 2\pi t)$ induces a compactification. This is equivalent to the one-point -compactification of the U- space X.

Theorem 4.13. If X is a Hausdorff locally compact U- space, then X_∞ is also Hausdorff U- space.
Proof: To prove this theorem it is enough to show that for any point x of X there exist two U-open sets G and H of X_∞ such that $x\in G$, $\infty \in H$ and $G \cap H = \Phi$. Let $x\in X$, then there exists a U-open set G such that $x\in G$ and $\overline{G}$ is a compact U- space of X. Let H = Y - $\overline{G}$, then G and H are U- open sets of Y and $x\in G$, $\infty \in H$ and $G \cap H = \Phi$.

Definition 4.15. [7](pp. 134). Let A and B be two U- spaces and h: $A \to B$ is a U- continuous, open and one-to-one map. Then h(A) is a U-homeomorphic subspace of A contained in B. Here A is called U-imbedded in B with U-imbedding h.
If A and h(A) are identified with each other, then A is a U-subspace of B.

Definition 4.16. A compact Hausdorff U- space Y is equivalently called a compactification (see above) of a U –space X if there is a U- imbedding h: $X \to Y$ such that h(X) is U- dense in Y. i. e. if Y is an extension U- space of h(X).

We conclude the paper with generalization of a theorem in Munkres [9](pp. 238).

Theorem 4.14. Let X be a U- spce. Let h: $X \to Z$ be a U-imbedding of X in the compact Hausdorff U-space Z. Then there exists a corresponding compactification Y of U- space X; which has the property that there is a U-imbedding H: $Y \rightarrow Z$ that equals h on X.

Proof: Given h, let X_o denote the U- subspace h(X) of Z, and Y_o denote its closure in Z. Then Y_o is a compact Hausdorff U- space and $\overline{X_o} = Y_o$; therefore, Y_o is an compactification of X_o.

We now construct a U- space Y containing X such that the pair (X, Y) is U-homeomorphic to the pair (X_o, Y_o). Let us choose a set A disjoint from X that is in bijective correspondence with the set $Y_o - X_o$ under some map k: $A \rightarrow Y_o - X_o$. Define $Y = X \cup A$, and define a bijective correspondence H: $Y \rightarrow Y_o$ by the rule H(x) = h(x) for $x \in X$, $H(\alpha) = k(\alpha)$ for $\alpha \in A$.

Make Y into a U- space by declaring V to be U- open in Y if and only if H(V) is U- open in Y_o. The map H is automatically a U-homeomorphism; and the U-space X is a U-subspace of Y because H equals the U-homeomorphism h when restricted to the U-subspace X of Y. By expanding the range of H, we obtain the required U-imbedding of Y into Z.

Acknowledgements. The second author is thankful to the Rajshahi University for the facilities made available to him.

Also the authors are grateful to the referees for their valuable comments to improve the presentation of the paper.

REFERENCES

1. D. Andrijevic, On b-open sets, *Mat. Vesnik*, 48 (1996) 59-64.
2. R. Devi, S. Sampathkumar and M. Caldas, On supra α open sets and Sα-continuous functions, *General Mathematics*, 16(2) (2008) 77-84.
3. B. Garai and C. Bandyopadhyay, On irresolvable spaces, Bull. Cal. Math. Soc., 95 (2) (2003) 107.
4. A.M. Gleason, Projective topological spaces, *Illinois J. Math.*, 2 (1958) 482-489.
5. T. Indira and K. Rekha, Applications of *b-open Sets and **b-open Sets in Topological Spaces, *Annals of Pure and Applied Mathematics*, 1(1) (2012) 44-56.
6. S. Majumdar and M. Mitra, On some generalizations of open sets and closed sets, *Journal Tri. Math. Soc.*. 11(2009) 100-113.
7. S. Majumdar and N. Akhter, Topology (Bengali), Adhuna Prakashan, 2009.
8. S. Mashhour, A. A. Allam, F. S. Mahmoud and F. H. Khedr, On supratopological spaces, *Indian J. Pure Appl, Math*, 14(4) (1983) 502-510.
9. J. R. Munkres, *Topology*, Prentice Hall of India Private Limited, New Delhi-110001, 2006.
10. O. Njasted, On some classes of nearly open sets, *Pacific J. Math.*, 15 (1965) 961-970.
11. V.Popa and T.Noiri, On the points of continuity and discontinuity, *Bull U.P.G. PL. Ser. Mat. Fiz.*, LRI, 1 (2000) 95.
12. O. R. Sayed and Takashi Noiri, On supra b-open sets and supra b-continuity on topological spaces, *European Journal of Pure and Applied Mathematics*, 3(2) (2010) 295-302.
13. N. V.Velicko, H-closed topological spaces, *Amer. Math. Soc. Transl.*, 78 (1968) 103-118.