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On Matrices Over Semirings

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Abstract. In this paper, matrices over semirings are investigated. This is done by introducing some examples of semirings and presenting some results on regular and invertible matrix semirings. These include conditions for regularity and invertibility of matrices over semirings as generalization of corresponding results on matrices over rings. Examples and results are illustrated by computing using MATLAB.

Keywords: Idempotent, Additively commutative semiring, Regular

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1. Introduction

The notion of semiring was first introduced by H.S. Vandiver in 1934. H.S. Vandiver introduced an algebraic system, which consists of non empty set S with two binary operations addition (+) and multiplication (.). The system (S; +, .) satisfies both distributive laws but does not satisfy cancellation law of addition. The system he constructed was ring like but not exactly a ring. Vandiver called this system a 'Semiring'.
The study of matrices over general semirings has a long history. In 1964, Rutherford [3] gave a proof of Cayley –Hamilton theorem for a commutative semiring avoiding the use of determinants. Since then, a number of works on theory of matrices over semirings were published [1, 12]. In 1999, J S Golan described semirings and matrices over semirings in his work [5] comprehensively. The techniques of matrices have important applications in optimization theory, models of discrete event network and graph theory.
Luce [10] characterized the invertible matrices over a Boolean algebra of at least two elements. Rutherford [2] has introduced that a square matrix over a Boolean algebra of 2 elements is invertible. Additively inverse semirings are studied by Karvellas [12]. Kaplansky [4], Petrich [9], Goodearl [6], Reutenauer [1], Fang [8] have studied semiring.

2. Preliminaries

In this section, we present some definitions and examples of semiring. MATLAB function scripts are not presented in this paper but inputs and outputs from the computer are given.

Definition 2.1. Let S be non empty set with two binary operations **+** and **.** . Then the algebraic structure (S; +, .) is called **a *semiring*** iff

$\forall a,b,c \in S$;

(i) (S; +) is a semigroup
(ii) (S; .) is a semigroup
(iii) a. (b+c) = a.b + a.c and (b+c).a = b.a + c.a.

Example 2.1(a). (L = {5, 10, 20, 25, 40, 50, 100, 200}; | , +, .) is a semiring, where

a + b = lcm{a , b}, a.b = gcd{a , b}.

The MATLAB function scripts are not shown. Outputs are presented below.

```
>> A=[5 10  20 25 40 50 100 200];

A =
     5    10    20    25    40    50   100   200
>> join(A)
ans =
```

+	5	10	20	25	40	50	100	200
5	5	10	20	25	40	50	100	200
10	10	10	20	50	40	50	100	200
20	20	20	20	100	40	100	100	200
25	25	50	100	25	200	50	100	200
40	40	40	40	200	40	200	200	200
50	50	50	100	50	200	50	100	200
100	100	100	100	100	200	100	100	200
200	200	200	200	200	200	200	200	200

```
>> meet(A)
ans =
```

.	5	10	20	25	40	50	100	200
5	5	5	5	5	5	5	5	5
10	5	10	10	5	10	10	10	10
20	5	10	20	5	20	10	20	20
25	5	5	5	25	5	25	25	25
40	5	10	20	5	40	10	20	40
50	5	10	10	25	10	50	50	50
100	5	10	20	25	20	50	100	100
200	5	10	20	25	40	50	100	200

Definition 2.2. Let (S; +, .) be a semiring**.** Then S is called

(i) ***additively commutative*** iff $\forall x, y \in S, x + y = y + x.$

(ii) ***multiplicatively commutative*** iff $\forall x, y \in S,\ x.y = y.x.$

(S; +, .) is called ***a commutative semiring*** iff both (i) and (ii) hold.

Example 2.2 (a). Every bounded distributive lattice is a commutative semiring under join and meet.

Definition 2.3. Let (S; +, .) be a semiring**.** Then an element $0 \in S$ is called ***zero of S*** iff $\forall x \in S$,

$$x + 0 = x = 0 + x \text{ and } x.0 = 0 = 0.x.$$

Example 2.3 (a). Consider the set of positive integers $\mathbb{Z}^+$ with the operations a + b = lcm{a , b} and a . b = ab. Then $(\mathbb{Z}^+;+,.)$ is a semiring with zero element 1, but 1 is not zero, since 1.a = a.1 = a $\neq 1$ for any $a \in \mathbb{Z}^+$ and $a \neq 1$.

Definition 2.4. Let (S; +, .) be a semiring**.** Then an element $1 \in S$ is called ***identity of S*** iff $\forall x \in S$, $x.1 = x = 1.x$.

Example 2.4 (a). Let $X \neq \phi$ and P(X) is power set of X. + and . are defined by $A + B = A \cup B$ and A.B $= A \cap B; \forall A, B \in P(X)$. Then (P(X); +, .) is a semiring, where ϕ and X are zero and identity of P(X) respectively.

Definition 2.5. Let (S; +, .) be a commutative semiring with zero (0) and identity (1) . Then (S; +, .) is called ***idempotent semiring*** iff $\forall x \in S$,

$$x + x = x = xx.$$

Example 2.5 (a). (I =[0,1]; +, .) is a ***an idempotent semiring***, where order in [0,1] is usual $\leq$ and + and . are defined as follows:

$$a + b = \max\{a , b\},\ a.b = \min\{a , b\}.$$

Proposition 2.6. Let (S; +, .) be an idempotent semiring with zero (0) and identity (1). Then

(a) $\forall x, y \in S$, $x + y = 0 \Rightarrow x = 0 = y$

(b) $\forall x, y \in S$, $xy = 1 \Rightarrow x = 1 = y.$

Proof: (a) By the definition of idempotent semiring $\forall x \in S$,

$$x + x = x = x^2.$$

Let

$$x + y = 0 \qquad \text{... (i)}$$

Putting x for y in (i)

$$x + x = 0$$
$$\Rightarrow x = 0$$

Putting y for x in (i)

$$y + y = 0$$
$$\Rightarrow y = 0$$

Hence $x = 0 = y$

(b) Let

$$xy = 1. \qquad \ldots\text{(ii)}$$

Putting x for y in (ii)

$$xx=1$$
$$\Rightarrow x^2 = 1$$
$$\Rightarrow x = 1$$

Putting y for x in (ii)

$$yy = 1$$
$$\Rightarrow y^2 = 1$$
$$\Rightarrow y = 1$$

Hence $x = 1 = y$ Δ

3. **Matrices over semirings**

Throughout this section (S; +, .) is an additively commutative semiring with zero (0) and identity (1) ($1 \neq 0$). n is positive integer and $M_n(S)$ is set of all $n \times n$ matrices over S.

Proposition 3.1. For any semiring S, ($M_n(S)$; +, .) is a semiring. Further if S is additively commutative then $M_n(S)$ is additive commutative. If S has zero then $M_n(S)$ has zero. If S has zero as well as identity then $M_n(S)$ has identity. In $M_n(S)$, + and . are defined by $[a_{ij}]+[b_{ij}]=[a_{ij}+b_{ij}]$ and $[a_{ij}][b_{ij}]=\sum_{k=1}^{n} a_{ik}.b_{kj}$.

Proof: We know

$$A,\ B \in M_n(S) \Rightarrow A+B \in M_n(S).$$

Again + is associative on the set of matrices, so for all $A,\ B,\ C \in M_n(S)$,
A + (B + C) = (A + B) + C.

Therefore ($M_n(S)$; +) is semigroup.
Similarly

$$A,\ B \in M_n(S) \Rightarrow AB \in M_n(S).$$

Again . is associative on the set of matrices, so for all $A,\ B,\ C \in M_n(S)$,

A. (B.C) = (A.B).C.

Therefore ($M_n(S)$; .) is semigroup.

Moreover for all $A,\ B,\ C \in M_n(S)$,

A.(B + C) = A.B + A.C
and (A + B).C = A.C + B.C.

Therefore ($M_n(S)$; +, .) is a semiring.
Again + is commutative on the set of matrices i.e. A+B = B+A.
Since $0,1 \in S$, so $0 \in M_n(S)$ and $I \in M_n(S)$.
We have A+ 0 = A= 0 + A and AI = A = IA.

Hence ($M_n(S)$; +, .) is additively commutative semiring with zero and identity. Δ

Example 3.1(a). (B = {0,1}, +, .) is a commutative semiring, where + and . are defined as in [7]. Then $M_n(B)$, set of all $n\times n$ matrices over B is an additively commutative semiring.

Here $|M_n(B)| == 2^{n^2}$.

For example $M_2(B)$ has $2^{2^2} = 16$ elements:

```
>> MatList
```

$O = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$, $C = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$, $D = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$, $E = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$, $F = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$,

$G = \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix}$, $H = \begin{pmatrix} 0 & 0 \\ 1 & 1 \end{pmatrix}$, $P = \begin{pmatrix} 1 & 0 \\ 1 & 0 \end{pmatrix}$, $Q = \begin{pmatrix} 0 & 1 \\ 0 & 1 \end{pmatrix}$, $I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$,

$S = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$, $T = \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix}$, $U = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}$, $V = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$, $W = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}$, $X = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$

generated by MATLAB function script. The addition table also generated by MATLAB function scripts is shown below:

```
>> MatSum
MatSum =
```

+	O	C	D	E	F	G	H	P	Q	I	S	T	U	V	W	X
O	O	C	D	E	F	G	H	P	Q	I	S	T	U	V	W	X
C	C	C	G	P	I	G	U	P	V	I	W	X	U	V	W	X
D	D	G	D	S	Q	G	T	W	Q	V	S	T	X	V	W	X
E	E	P	S	E	H	W	H	P	T	U	S	T	U	X	W	X
F	F	I	Q	H	F	V	H	U	Q	I	T	T	U	V	X	X
G	G	G	G	W	V	G	X	W	V	V	W	X	X	V	W	X
H	H	U	T	H	H	X	H	U	T	U	T	T	U	X	X	X
P	P	P	W	P	U	W	U	P	X	U	W	X	U	X	W	X
Q	Q	V	Q	T	Q	V	T	X	Q	V	T	T	X	V	X	X
I	I	I	V	U	I	V	U	U	V	I	X	X	U	V	X	X
S	S	W	S	S	T	W	T	W	T	X	S	T	X	X	W	X
T	T	X	T	T	T	X	T	X	T	X	T	T	X	X	X	X
U	U	U	X	U	U	X	U	U	X	U	X	X	U	X	X	X
V	V	V	V	X	V	V	X	X	V	V	X	X	X	V	X	X
W	W	W	W	W	X	W	X	W	X	X	W	X	X	X	W	X
X	X	X	X	X	X	X	X	X	X	X	X	X	X	X	X	X

Her**e** zero, O = $\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$ and identity, I = $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$.

Proposition 3.2. $M_n(\mathbb{Z}_0^+)$, $M_n(\mathbb{Q}_0^+)$, $M_n(\mathbb{R}_0^+)$ are additively commutative semiring with zero under usual addition and multiplication.These are not multiplicatively commutative. Also, none of them are rings.

Proof : $\forall A, B, C \in M_n(\mathbb{Z}_0^+)$

Clearly $M_n(\mathbb{Z}_0^+)$ is an additively commutative semiring with zero.

Let us show by an example $M_n(\mathbb{Z}_0^+)$ is not multiplicatively commutative:

Let

$$A = \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix} \text{ and } B = \begin{pmatrix} 3 & 1 \\ 0 & 2 \end{pmatrix}$$

Now

$$AB = \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 3 & 1 \\ 0 & 2 \end{pmatrix} = \begin{pmatrix} 3 & 5 \\ 0 & 2 \end{pmatrix}$$

and $$BA = \begin{pmatrix} 3 & 1 \\ 0 & 2 \end{pmatrix} \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 3 & 7 \\ 0 & 2 \end{pmatrix}$$

Therefore $AB \neq BA$

Hence $M_n(\mathbb{Z}_0^+)$ is not multiplicatively commutative.

Last Part:

We have

$$A = \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix} \in M_n(\mathbb{Z}_0^+)$$

Then

$$-A = \begin{pmatrix} -1 & -2 \\ 0 & -1 \end{pmatrix} \notin M_n(\mathbb{Z}_0^+)$$

Such that

A + (-A) = 0.

Hence $M_n(\mathbb{Z}_0^+)$ is not a ring.

The case for $M_n(\mathbb{Q}_0^+)$ and $M_n(\mathbb{R}_0^+)$ are similar. Δ

Definition 3.3. Let (S; +, .) be a semiring. Then S is called *a* ***regular semiring*** iff $\forall x \in S, \exists y \in S$ such that $xyx = x$.

Proposition 3.4. Let (S; +, .) be an additively commutative semiring with zero (0) and n a positive integer. If $M_n(S)$ is a regular semiring, then so is S.

Proof: Let $M_n(S)$ be a regular semiring.

Then $\forall A \in M_n(S), \exists B \in M_n(S)$ such that $ABA = A$.

Define $A \in M_n(S)$ by $A(i,j) = \begin{cases} 0, \ \forall i, j \in \mathbb{N} \text{ with } (i,j) \neq (1,1) \\ a, \quad \text{for } (i,j) = (1,1) \end{cases}$

Then $A = \begin{pmatrix} a & 0 & 0........0 \\ 0 & 0 & 0........0 \\ 0 & 0 & 0........0 \\ .... \\ 0 & 0 & 0........0 \end{pmatrix}$

For this A, let

$$B = \begin{pmatrix} B(1,1) & B(1,2) & B(1,3).........B(1,n) \\ B(2,1) & B(2,2) & B(2,3)........B(2,n) \\ B(3,1) & B(3,2) & B(3,3).........B(3,n) \\ .... \\ B(n,1) & B(n,2) & B(n,3)........B(n,n) \end{pmatrix}$$

be such that *ABA*=*A*.

Now $AB = \begin{pmatrix} a.B(1,1) & 0 & .... & 0 \\ 0 & 0 & .... & 0 \\ 0 & 0 & .... & 0 \\ \multicolumn{4}{c}{...} \\ 0 & 0 & .... & 0 \end{pmatrix}$

$\therefore ABA = \begin{pmatrix} a.B(1,1).a & 0 & ....0 \\ 0 & 0 & ....0 \\ 0 & 0 & ....0 \\ \multicolumn{3}{c}{...} \\ 0 & 0 & .... 0 \end{pmatrix}$

Now $ABA = A \Rightarrow a.B(1,1).a = a.$ Δ

Remark 3.5. The converse of the above Proposition 3.4 is not necessarily true for n=2.

Let us show it by an example:
(S = {1, 2, 3, 6}; | , +, .) is a regular semiring, where a + b = lcm{a , b}, a.b = gcd{a , b}; $\forall a, b \in S$.

But $M_2(S)$ is not regular semiring.

Let $A = \begin{pmatrix} 1 & 2 \\ 3 & 6 \end{pmatrix} \in M_2(S)$

For this A, let $B=\begin{pmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{pmatrix}$ such that ABA=A.

Now $ABA=\begin{pmatrix} 1 & 2 \\ 3 & 6 \end{pmatrix}\begin{pmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{pmatrix}\begin{pmatrix} 1 & 2 \\ 3 & 6 \end{pmatrix}$

$$=\begin{pmatrix} 1.b_{11}+2.b_{21} & 1.b_{12}+2.b_{22} \\ 3.b_{11}+6.b_{21} & 3.b_{12}+6.b_{22} \end{pmatrix}\begin{pmatrix} 1 & 2 \\ 3 & 6 \end{pmatrix}$$

$$=\begin{pmatrix} 1.(1.b_{11}+2.b_{21})+3.(1.b_{12}+2.b_{22}) & 2.(1.b_{11}+2.b_{21})+6.(1.b_{12}+2.b_{22}) \\ 1.(3.b_{11}+6.b_{21})+3.(3.b_{12}+6.b_{22}) & 2.(3.b_{11}+6.b_{21})+6.(3.b_{12}+6.b_{22}) \end{pmatrix}$$

$$=\begin{pmatrix} 1.(1.b_{11}+2.b_{21})+3.(1.b_{12}+2.b_{22}) & 2.(1.b_{11}+2.b_{21})+(1.b_{12}+2.b_{22}) \\ 1.(3.b_{11}+6.b_{21})+3.(3.b_{12}+6.b_{22}) & 2.(3.b_{11}+6.b_{21})+(3.b_{12}+6.b_{22}) \end{pmatrix}$$

A=ABA follows that

$$1.(1.b_{11}+2.b_{21})+3.(1.b_{12}+2.b_{22})=1. \qquad \ldots\text{(i)}$$

$$2.(3.b_{11}+6.b_{21})+(3.b_{12}+6.b_{22})=6. \qquad \ldots\text{(ii)}$$

From (i) we get

$$3.(2.b_{22})=1 \Rightarrow b_{22}=1$$

From (ii), clearly

$$2.(3.b_{11}+6.b_{21}) \le 2$$

From (ii) we get

$$(3.b_{12}+6.b_{22})=6$$

But $b_{22}=1$, so

$$6=3.b_{12}+6.1=3.b_{12}+1=3.b_{12}\le 3,$$

which is a contradiction.

So $M_2(S)$ is not regular.

Example 3.7. (B = {0,1}; +, .) is a commutative semiring with zero, where + and . are defined as in [7]. Then $M_2(B)$, set of all 2×2 matrices over B is regular semiring.

Remark 3.8. (B = {0,1}; +, .) is a commutative semiring, where + and . are defined as in [7]. Then $M_2(B)$, set of all 2×2 matrices over B is not idempotent semiring.

From Example 3.1(a), the Matlist generated by MATLAB function script. The multiplication table also generated by MATLAB function scripts is shown below:

```
>> Matmult

Matmult =
```

.	O	C	D	E	F	G	H	P	Q	I	S	T	U	V	W	X
O	O	O	O	O	O	O	O	O	O	O	O	O	O	O	O	O
C	O	C	D	O	O	G	O	C	D	C	D	D	C	G	G	G
D	O	O	O	C	D	O	G	C	D	D	C	G	G	D	C	G
E	O	E	F	O	O	H	O	E	F	E	F	F	E	H	H	H
F	O	O	O	E	F	O	H	E	F	F	E	H	H	F	E	H
G	O	C	D	C	D	G	G	C	D	G	G	G	G	G	G	G
H	O	E	F	E	F	H	H	E	F	H	H	H	H	H	H	H
P	O	P	Q	O	O	X	O	P	Q	P	Q	Q	P	X	X	X
Q	O	O	O	P	Q	O	X	P	Q	Q	P	X	X	Q	P	X
I	O	C	D	E	F	G	H	P	Q	I	S	T	U	V	W	X
S	O	E	F	C	D	H	G	P	Q	S	I	V	W	T	U	X
T	O	E	F	P	Q	H	X	P	Q	T	U	X	X	T	U	X
U	O	P	Q	E	F	X	H	P	Q	U	T	T	U	X	X	X
V	O	C	D	P	Q	G	X	P	Q	V	W	X	X	V	W	X
W	O	P	Q	C	D	X	G	P	Q	W	V	V	W	X	X	X
X	O	P	Q	P	Q	X	X	P	Q	X	X	X	X	X	X	X

From Matmult Table we see that

$$D^2 = 0 \neq D,\ E^2 = 0 \neq E,\ S^2 = I \neq S,\ T^2 = X \neq T,\ W^2 = X \neq W.$$

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