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Djoudi's Common Fixed Point Theorem on Compatible Mappings of Type (P)

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Abstract. The aim of this paper is to prove a common fixed point theorem which generalizes the result of Djoudi by weaker conditions such as compatible mappings of type (P) and associated sequence. We constructed one example in which the mappings are only compatible mappings of type (P) but not any one of compatible, compatible mappings of type (A), compatible mappings of type (B).

Keywords: Fixed point, Self-maps, Compatible mappings of type (P), Associated sequence.

AMS Mathematics Subject Classification (2010): 54H25

1. Introduction

Two self maps S and T on a metric space (X,d) are said to be commute if ST= TS. According to Jungck [1], Two self maps S and T of a metric space (X,d) are said to be compatible mappings if $\lim_{n\to\infty} d(STx_n, TSx_n) = 0$, whenever $<x_n>$ is a sequence in X such that $\lim_{n\to\infty} Sx_n = \lim_{n\to\infty} Tx_n = t$ for some $t \in X$.

From Jungck and others [2], two self maps S and T of a metric space (X, d) are said to be compatible mappings of type(A), if $\lim_{n\to\infty} d(STx_n, TTx_n) = 0$ and $\lim_{n\to\infty} d(TSx_n, SSx_n) = 0$ whenever $<x_n>$ is a sequence in X such that $\lim_{n\to\infty} Sx_n = \lim_{n\to\infty} Tx_n = t$ for some $t \in X$.

By Pathak and others [3], two self maps S and T of a metric space (X,d) are said to be compatible mappings of type (P), if $\lim_{n\to\infty} d(SSx_n, TT_n) = 0$, when ever $<x_n>$ is a sequence in X such that $\lim_{n\to\infty} Sx_n = \lim_{n\to\infty} Tx_n = t$ for some $t \in X$.

In view of Pathak and others [4], two self maps S and T of a metric space (X,d) are said to be weak compatible mappings of type(A), if $\lim_{n\to\infty} d(STx_n, TTx_n) \le \lim_{n\to\infty} d(TSx_n, TTx_n)$

and $\lim_{n\to\infty} d(TSx_n,SSx_n) \le \lim_{n\to\infty} d(STx_n,SSx_n)$ whenever $<x_n>$ is a sequence in X such that $\lim_{n\to\infty} Sx_n = \lim_{n\to\infty} Tx_n = t$ for some $t\in X$.

According to Pathak and Khan [5], two self maps S and T of a metric space (X,d) are said to be compatible mappings of type(B), if

$\lim_{n\to\infty} d(STx_n,TTx_n) \le \frac{1}{2}[\lim_{n\to\infty} d(STx_n,St) + \lim_{n\to\infty} d(St,SSx_n)]$ and

$\lim_{n\to\infty} d(TSx_n,SSx_n) \le \frac{1}{2}[\lim_{n\to\infty} d(TSx_n,Tt) + \lim_{n\to\infty} d(Tt,TTx_n)]$ whenever $<x_n>$ is a sequence in X such that $\lim_{n\to\infty} Sx_n = \lim_{n\to\infty} Tx_n = t$ for some $t\in X$.

Definition 1.1: A function $\phi: [0,\infty) \to [0,\infty)$ is said to be a *contractive modulus* if $\phi(0)=0$ and $\phi(t) < t$ for $t > 0$.

Eg: $\phi: [0,\infty) \to [0,\infty)$ defined by $\phi(t) = ct$, where $0\le c<1$ is a contractive modulus.

Definition 1.2: A real valued function ϕ defined on $X \subseteq R$ is said to be *Upper semi continuous* if $\lim_{n\to\infty} \phi(t_n) \le \phi(t)$, for every sequence $<t_n>\in X$ with $t_n \to t$ as $n\to\infty$. Every continuous function is upper semi continuous but not conversely.

2. A common fixed point theorem

Let R_+ be the set of non negative real numbers and let $\varphi: R^5_+ \to R_+$ be a function satisfying the following conditions:

φ is upper semi continuous in each coordinate variable and non decreasing.

$\phi(t) = \max\{\varphi(0,t,0,0,t), \varphi(t,0,0,t,t), \varphi(t,t,t,2t,0), \varphi(0,0,t,t,0)\} < t$ for any $t > 0$.

The following is the theorem proved by Djoudi [6].

2.1 Theorem

Let I, J, S and T be mappings from a complete metric space (X,d) into itself satisfying the conditions

(2.1.1) $S(X)\subset J(X)$ and $T(X)\subset I(X)$

(2.1.2) $d(Sx,Ty)\le\max\{\varphi(d(Ix,Jy),d(Ix,Sx),d(Jy,Ty), d(Ix,Ty),d(Jy,Sx))\}$ for all $x,y\in X$.

(2.1.3) one of S,I,T and J is continuous

(2.1.4) the pairs (S,I) and (T,J) are compatible mappings of type(B)

Then S,I,T and J have a unique common fixed point z. Furthermore z is the unique common fixed point of both mappings.

2.2 Associated Sequence

Suppose S,I,T and J are self maps of a metric space (X,d) satisfying the condition $S(X)\subseteq J(X)$ and $T(X)\subseteq I(X)$. Then for any $x_0\in X$, $Sx_0\in S(X)$ so that there is a $x_1\in X$ with $Sx_0 = Jx_1$. Now $Tx_1\in TX$ and hence there is $x_2\in X$ with $Tx_1= Ix_2$. Repeating this process to each $x_0\in X$, we get a sequence $<x_n>$ in X such that $Sx_{2n}= Jx_{2n+1}$ and $Tx_{2n+1} = Ix_{2n+2}$ for $n \ge 0$. We shall call this sequence as an "*Associated sequence of x_0* "relative to the four elf maps S,I,T and J.

Lemma 2.3. [6] Suppose S,I,T and J are three self maps of a metric space (X, d) for which the conditions (a). S(X)$\subseteq$J(X) and T(X)$\subseteq$I(X) and satisfying the condition

$$\left[d(Sx,Ty)\right] \le \phi\left\{d(Ix,Jy), d(Ix,Sx), d(Jy,Ty), d(Ix,Ty), d(Jy,Sx)\right\}.$$

Further if (X,d) is a complete metric space then for any $x_0 \in X$ and for any of its associated sequence $<x_n>$ relative to four self maps, the sequence $Sx_0, Tx_1, Sx_2, Tx_3, \ldots Sx_{2n}, Tx_{2n+1}, \ldots$ converges to some point $z \in X$ (1)

The converse of the lemma is not true. That is, suppose S,I, T and J are self maps of a metric space (X,d) satisfying the conditions mentioned in the lemma and even for each associated sequence $<x_n>$ of x_0, the sequence in (1) converges, the metric space (X,d) need not be complete.

Example 2.4. Let X = [0,1/2) with d(x,y) = $|x-y|$ Define self maps S, I, T and J of X by Ix=Jx = $\frac{1}{2}-x$ if x $\in$ [0,1/2) and

$$Sx = Tx = \begin{cases} \frac{1}{4} & if \quad x \in \left[0, \frac{1}{4}\right] \\ \frac{1}{3} & if \; x \in \left(\frac{1}{4}, \frac{1}{2}\right) \end{cases}$$

Clearly S(X)$\subseteq$ J(X) and T(X) $\subseteq$I(X). The associated sequence $Sx_0, Tx_1, Sx_2, Tx_3, \ldots, Sx_{2n}, Tx_{2n+1}, \ldots$, converges to the point 1/4 . But X is not a complete metric space.

3. Main Result

Theorem 3.1. Let S,I,T, and J are self maps of a metric space (X,d) satisfying the conditions

(3.1.1). S(X)$\subseteq$ J(X) and T(X) $\subseteq$I(X)

(3.1.2) $\left[d(Sx,Ty)\right] \le \phi\left\{d(Ix,Jy), d(Ix,Sx), d(Jy,Ty), d(Ix,Ty), d(Jy,Sx)\right\}$ for all x,y in X.

(3.1.3) One of S,I,T, and J is continuous

(3.1.4) (S,I) and (T,J) are compatible mappings of type(P).

(3.1.5) The sequence $Sx_0, Tx_1, Sx_2, Tx_3, .., Sx_{2n}, Tx_{2n+1}, \ldots$, converges to $z \in X$.

Then S, I, T and J have a unique common fixed point in X.

Proof: From condition (3.1.5) $Sx_{2n} \to z$ and $Tx_{2n+1} \to z$ as $n \to \infty$

Suppose S is continuous then $SSx_{2n} \to Sz$, $SIx_{2n} \to Sz$ as $n \to \infty$.

Since (S, I) is compatible mappings of type(P)

$\lim_{n\to\infty} d(SSx_{2n}, IIx_{2n}) = 0$. This gives $SSx_{2n} \to Az$ as $n \to \infty$

$\lim_{n\to\infty} SSx_{2n} = \lim_{n\to\infty} IIx_{2n} = Sz.$

Put x = Ix_{2n} y = x_{2n+1} in (3.1.2)

$[d(SIx_{2n}, Tx_{2n+1})] \leq \phi\{d(IIx_{2n}, Jx_{2n+1}), d(IIx_{2n}, SIx_{2n}),$

$d(Jx_{2n+1}, Tx_{2n+1}), d(IIx_{2n}, Tx_{2n+1}), d(Jx_{2n+1}, SIx_{2n})\}$

Letting $n \to \infty$

$[d(Sz,z)] \leq \phi\{d(Sz,z), d(Sz,Sz), d(z,z), d(Sz,z), d(z,Sz)\}$

$d(Az,z) \leq \phi\{d(Sz,z), 0, 0, d(Sz,z), d(z,Sz)\}$

$d(Az,z) \leq \phi d(Sz,z) < d(Sz.z)$, a contradiction $if\ Sz \neq z$

Therefore $Sz = z.$

Since S(X)$\subseteq$J(X) implies there exists u$\in$ X such that z=Sz=Ju

To prove $Tu = z\, in$ (3.1.2)

Put x = x_{2n}, y = u

$[d(Sx_{2n}, Tu)] \leq \phi\{d(Ix_{2n}, Ju), d(Ix_{2n}, Sx_{2n}), d(Ju, Tu), d(Ix_{2n}, Tu), d(Ju, Sx_{2n})\}$

Letting $n \to \infty$

$[d(z,Tu)] \leq \phi\{d(z,z), d(z,z), d(z,Tu), d(z,Tu), d(z,z)\}$

$d(z,Tu) \leq \phi\{0, 0, d(z,Tu), d(z,Tu), 0\}$

$d(z,Tu) \leq \phi d(z,Tu) < d(z.Tu)$, a controduction $if\ Tu \neq z$

$[d(Tu,z)] = 0\, or\, Tu = z$

Therefore, $Ju = Tu = z$

Since (T, J) is compatible mappings of type(P), w e have

$d(TTu, JJu) = 0$.This gives $d(Bz, Tz) = 0$ or $Tz = Jz$

To prove $Tz = z.$

Put x = x_{2n}, y = z in (3.1.2)

$[d(Sx_{2n}, Tz)] \leq \phi\{d(Ix_{2n}, Jz), d(Ix_{2n}, Sx_{2n}), d(Jz, Tz), d(Ix_{2n}, Tz), d(Jz, Sx_{2n})\}$

Letting $n \to \infty$

$[d(z,Tz)] \leq \phi\{d(z,Tz), d(z,z), d(Tz,Tz), d(z,Tz), d(Tz,z)\}$

$[d(z,Tz)] \leq \phi\{d(z,Tz), 0, 0, d(z,Tz), d(Tz,z)\}$

$d(z,Tz) \leq \phi d(z,Tz) < d(z.Tz)$, a contradiction if $Tz \neq z$

$[d(Tz,z)]$=0.

Therefore $Tz = z.$

Therefore $Jz = Tz = z.$

Since $\mathrm{T}(\mathrm{X}) \subseteq I(\mathrm{X})$ implies there exists $v \in \mathrm{X}$ such that $\mathrm{z} = Tz = Iv.$

To prove $Sv = z.$

Put $\mathrm{x} = v, \mathrm{y} = z$ *in* (3.1.2)

$[d(Sv,Tz)] \le \phi\{d(Iv, Jz), d(Iv, Sv), d(Jz,Tz), d(Iv,Tz), d(Jz, Sv)\}$

Letting $n \to \infty$

$[d(Sv, z)] \le \phi\{d(z, z), d(z, Sv), d(z, z), d(z, z), d(z, Sv)\}$

$[d(Sv, z)] \le \phi\{0, d(z, Sv), 0, 0, d(z, Sv)\}$

$d(Sv, z) \le \phi d(z, Sv) < d(z.Sv)$, a contradiction if $Sv \ne z$

$[d(Tz, z)] = 0$ or $Sv = z.$

Therefore $z = Sv = Iv.$

Since (S, I) is compatible mappings of type(P) , w e have

$d(SSv, IIv) = 0.$ This gives $d(Sz, Iz) = 0$ or $Sz = Iz$

To prove $Sz = z.$

Put $\mathrm{x} = z, \mathrm{y} = z$ in (3.1.2)

$[d(Sz,Tz)] \le \phi\{d(Iz, Jz), d(Iz, Sz), d(Jz,Tz), d(Iz,Tz), d(Jz, Sz)\}$

$[d(Sz, z)] \le \phi\{d(Sz, z), 0, 0, d(Sz, z), d(Sz, z)\}$

$d(Sz, z) \le \phi d(Sz, z) < d(Sz, z)$, a contradiction if $Sz \ne z$

$[d(Sz, z)] = 0$ *or* $Sz = z.$

Therefore $z = Sz = Iz.$

Since Iz=Jz=Sz=Tz=z, we get z in a common fixed point of S,I,T and J. The uniqueness of the fixed point can be easily proved.

Example 3.2. Let X = [0,1/2) with d(x,y) = $|x - y|$. Define self maps S,I,T and J of X by Ix = Jx= $\frac{1}{2} - x$ if x∈ [0, 1/2) and

$$Sx = Tx = \begin{cases} \frac{1}{4} & if\ x \in \left[0, \frac{1}{4}\right] \\ \frac{1}{3} & if\ x \in \left(\frac{1}{4}, \frac{1}{2}\right) \end{cases}$$

Clearly the pairs (S,I) and (T,J) are not commutative, and it can be easily verified that the mappings are not compatible, compatible of type (A), weak compatible of type (A), and also not compatible of type (B) but they are compatible of type (P).

Also the condition (3.1.2) holds. We note that X is not a complete metric space and It is easy to prove that the associated sequence $Sx_0, Tx_1, Sx_2, Tx_3, .., Sx_{2n}, Tx_{2n+1} .....$, converges to the point 1/4 which is a common fixed point of S,I,T and J. In fact 1/4 is the unique common fixed point of S, I, T and J.

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