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S-I Epidemic Model Incorporating Social-Economic and Cultural Causes and Effects in Developing Countries

***Md. Nazmul Hasan*[1], *Harekrishna Das*[2] and *Prodip Kumar Ghosh*[3]**

[1]Department of Mathematics, Govt. M.M.College, Jessore, Bangladesh
[2]Department of Mathematics, Jessore Govt. City College, Jessore, Bangladesh
[3]Department of Mathematics, Jessore College, Jessore, Bangladesh

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Abstract. This paper represents our first attempt on modeling the effects of social, economical and cultural factors on the S-I epidemic diseases. The analysis of these models suggests that the lack of health cares and control of infected pregnant women aided the spread of the disease. Most of the findings are supported by systematic mathematical analysis including local and global stability results. The evaluations are carried out using numerical simulations with Matlab.

Keywords: epidemic diseases, global stability, sexual partnerships, disease-free equilibrium.

AMS Mathematics subject classification (2010): 92B05

1. Introduction

The models described incorporate to accommodate a farm or elaborate demographic consideration of the influence of age stages (juvenile and adult) on new infections. Modeling what is known, or suspected, about the importance of age of partners in the spread of the disease is a formidable problem, both conceptually and practically, but sexual gender has also an influence in the study of the se models about STD's. In populations without female dominance, both sexes must be incorporated in order to provide an appropriate representation of the population dynamics. The main difference between one-sex and two-sex epidemic models (except from the introduction of the second sex) is that the birth and survival schedules no longer are assumed to be constant, but depend on the size of the population, and its epidemic-sex composition.

It is not unknown that mathematical modeling has had an important role in the understanding of the HIV virus dynamics. We can mention some of the works, which are related to extensions in different aspects of the HIV dynamics. As a consequence of the above mentioned articles, a big number of models were studied trying to include or exclude biological facts into the mathematical translation, some of them are compartmental models [2], and some more recent works using delay differential models as in [4]. It is recommended [9] to get more references and interesting reading about mathematical models in HIV dynamics. Our research deals with the spread of STDs infections and in particular HIV epidemics. It is desirable to consider future work that could include the importance of the role of HIV virus dynamics into the spread of the STDs for the models described in our research.

2. Two sex S-I epidemic model

In this thesis we derive the mathematical model of new infections in consideration of sex structure. The infection will be spread through the formation of hetero sexual partnerships among active populations of males and females. The population will be disaggregated by sex and by whether or not infected, $S_l(t)\ (I = f, m)$ denote all sexually active susceptible individuals (female or male) and Il(t)denotes all sexually active infected individuals (female or male) with $N_l(t) = S_l(t) + I_l(t)(I = f, m)$ and $N(t) = S_f(t) + I_m(t)$ being the total population size. It is assumed that new sexually active susceptible (or infected respectively) female arrive at the rate $\sigma\beta_1 S_f\ (or\, \sigma\beta_2 I_f)$ where $\sigma\,(0 \le \sigma \le 1)$ is the probability to be a female newborn and $\beta_i\ (i = 1, 2)$ is the birth rate of susceptible (infected) individuals respectively. In the other hand, for new sexually active susceptible (or infected respectively) male, they arrive at the rate $(1-\sigma)\beta_i S_m(t)(or\ (1-\sigma)\beta_i S_f(t)$ respectively) with $i = 1, 2$. The parameter $C_{fm}(or\ C_{mf})$ denote the average number of contacts per partner female-male (or male-female), assumed $C_{fm} = C_{mf} = C$ then $CS_i(t)$ denotes the total number of contacts per unit of time of all susceptible (female or male) individuals at time t. In addition, we assume that only a fraction $\sigma\ (0 \le \sigma \le 1)$ of the new born are female and a fraction $\xi\ (0 \le \xi \le 1)$ describe the infected portion of birth from infected mothers. The parameter ξ could be considering like a control parameter that measure the quality of medical services of the infected mothers. If we consider the fraction $\zeta\ (0 \le \zeta \le 1)$ of the sexual contacts which don't use condom it develops into new cases of infection. We define the transmission rate for women as $C\sigma\zeta$ and for men as $C(1-\sigma)\zeta$, the expressions for the new cases of infection per unit time (incidence rate) in female and male population are $C\sigma\zeta S_f(I_m / N_m)$ and $C(1-\sigma)\zeta S_m(I_f / N_f)$ respectively. By natural death the individuals leave active populations at the rate μS_l (or μI_l) respectively and that infected Individuals leave their class at the death rate by infection $\gamma I_l\ (l = f, m)$. Then the model

$$\begin{cases} S_f' = \sigma\beta_1 S_f + \sigma(1-\xi)\beta_2 I_f - C\sigma\zeta S_f I_m / N_m - \mu S_f - mS_f N, \\ S_m' = (1-\sigma)\beta_1 S_f + (1-\sigma)(1-\xi)\beta_2 I_f - C(1-\sigma)\zeta S_m I_m / N_f - \mu S_m - mS_m N, \\ I_f' = \sigma\xi\beta_2 I_f + C\sigma\zeta S_f I_m / N_m - (\mu+\gamma) I_f - mI_f N \\ I_m' = (1-\sigma)\xi\beta_2 I_f + C(1-\sigma)\zeta S_m I_f / N_f - (\mu+\gamma) I_m - mI_m N. \end{cases} \quad (1)$$

And initial conditions $S_f(0) = S_f^0 > 0,\ I_f^0 > 0, S_m(0) = S_m^0 > 0\ and\ I_m(0) = I_m^0 > 0.$

3. Basic properties of solutions

The basic properties discussed here will be useful for next results about stability of the equilibria and persistence.

Theorem 1. For all $S_f^0, S_m^0, I_f^0, I_m^0 > 0$,there exists S_f, S_m, I_f, I_m which solves (1) and the initial conditions $S_f = S_f^0, S_m = S_m^0, I_f = I_f^0, I_m = I_m^0$

Proof: We define (Th. A2 [11]),

$$F_1(x) = \sigma\beta_1 S_f + \sigma(1-\upsilon)\beta_2 I_f - C\sigma\eta\frac{S_f I_m}{N_m} - \mu S_f - mS_f N$$

$$F_2(x) = (1-\sigma)\beta_1 S_f + (1-\sigma)(1-\upsilon)\beta_2 I_f - C(1-\sigma)\eta\frac{S_f I_m}{N_f} - \mu S_m - mS_m N$$

$$F_3(x) = \sigma\nu\beta_2 I_f + C\sigma\eta\frac{S_f I_m}{N_m} - (\mu+\gamma)I_f - mI_f N,$$

$$F_4(x) = (1-\sigma)\nu\beta_2 I_f + C(1-\sigma)\eta\frac{S_f I_f}{N_f} - (\mu+\gamma)I_m - mI_m N,$$

where $x = (S_f, S_m, I_f, I_m)$

By assumption (2) and the properties of continuity over operations, we have the continuity of F_i for all $i = 1, 2, 3, 4.$ further

$$\frac{\partial F_1}{\partial x_1} = (\sigma\beta_1 - \mu) - C\sigma\eta\frac{I_m}{N_m} - m(S_f + N)$$

$$\frac{\partial F_1}{\partial x_2} = C\sigma\eta\frac{S_f I_m}{N_m^2} - mS_f,$$

$$\frac{\partial F_1}{\partial x_3} = \sigma(1-\nu)\beta_2 - mS_f,$$

$$\frac{\partial F_1}{\partial x_4} = -C\sigma\eta\frac{S_f I_m}{N_m^2} - mS_f,$$

These partial derivatives exist and are continuous, in the same way the other partial derivatives are proved that exist and are continuous. In consequence F is locally Lipschitz continuous.

Let $x_1 = S_f = 0$ and $x_2 = S_m > 0, x_3 = I_f > 0, x_4 = I_m > 0,$ then

$F_1(x) = \sigma(1-\varepsilon)\beta_2 I_f > 0.$

Now let $x_2 = S_m = 0,$ and $x_1 = S_f > 0, x_3 = I_f > 0, x_4 = I_m > 0,$ them

$F_2(x) = (1-\sigma)(\beta_1 S_f + (1-\xi)\beta_2 I_F) > 0.$ Further, let $x_3 = I_f = 0,$ and

$x_1 = S_f > 0, x_2 = S_m > 0, x_4 = I_m$ them $F_3(x) = C\sigma\zeta \dfrac{S_f I_m}{N_m} > 0$ and finally let

$x_4 = I_m = 0$ and $x_1 = S_f > 0, x_2 = S_m > 0, x_3 = I_f > 0,$ then

$F_4(x) = (1-\sigma)[\zeta\beta_2 + C\zeta \dfrac{S_m}{N_f} > 0]I_f > 0.$

In consequence by Th.A2 in [11] every $x_0 = (S_f^0, S_m^0, I_f^0, I_m^0) \in R_+^4,$ there exists a unique solution of w $x' = F(x), x(0) = x_0$ with values en R_+^4 which is defined in some interval$_{(0,]}$ with $b \in (0,\infty].$ If $b < \infty$, them

$$\underset{0 \le t \le b}{SUP}(S_f(t) + S_m(t) I_f(t) + I_m(t)) = \infty.$$

Suppose that $b < \infty,$ them

$$N' = \beta_1 S_f + \beta_2 I_f \gamma(I_f + I_m) - \mu N - mN^2. \qquad (2)$$

If we take $\beta = \max\{\beta_1 \beta_2\}$, them we have $N' \le \beta N_f \le \beta N$,

in consequence $\dfrac{N'}{N_f} \le \beta$.

Integrating the above inequality

$$\text{In } N(t) \le \text{ In } N(0) + \beta t \ ,$$

it means $N(t) \le N(0)e^{\beta t}$. So $N(t)$ is bounded, a contradiction. In consequence $b = \infty$. Then the solutions of the system are positive and defined on $[0,\infty)$.

4. Boundedness of solutions

We will need the boundedness property to prove the global stability of some of the steady. We established the following result

Theorem 2. All the solution of the system (1) are bounded.

Proof: Set $\beta = \max\{\beta_1 . \beta_2\}.$

From equation (4.2), we have

$$N' = \beta_1 S_f + \beta_2 I_f - \gamma(I_f + I_m) - \mu N - mN^2$$
$$N' \le \beta N_f - mN^2,$$
$$\le \beta N - mN^2 \ ,$$

Claim: $N(t) \le M$ for all $t \ge 0,$ where $M = \max\{N(0), \dfrac{\beta}{m} + 1\}.$

We have two cases:

Case 1: If $M = \dfrac{\beta}{m} + 1$

Suppose the claim is not true: there exists a $t_1 > 0$ such that,

$$N(t_1) = \frac{\beta}{M} + 1, \qquad N(t) < \frac{\beta}{M} + 1, t < t_1, \qquad N'(t_1) \geq 0$$

$$N'(t_1) \leq \beta N(t_1) - mN^2(t_1),$$
$$= \beta\left(\frac{\beta}{m}+1\right) - m\left(\frac{\beta}{m}+1\right)^2 = -m\left(\frac{\beta}{m}+1\right) \quad ,$$

then $N'(t_1) < 0,$ it is a contradiction. In consequence, the claim is true.

Case 2: If $M = N(0)$.

Suppose the claim is not true: there exists a $t_1 > 0$ such that

$$N(t_1) = M, \qquad N'(t_1) \geq 0.$$
$$N'(t_1) \leq \beta N(t_1) - mN^2(t_1),$$
$$= \beta M - mM^2 = M(\beta - mM) \ ,$$

as $\dfrac{\beta}{M} < \dfrac{\beta}{m} + 1 \leq M$, them $\beta - mM < 0$ and $N'(t_1) < 0$, it is a contradiction. In consequence, the claim is true.

It means that $N(t) \leq M$ for all $t \geq 0$, them the theorem is proved.

5. Equilibria and stability

We first dimensionalize the system (1) with the following scaling:

$$\hat{t} = \mu t, \quad \hat{S}_f = \frac{S_f}{\mu}, \quad \hat{S}_m = \frac{S_m}{\mu}, \quad \hat{I}_f = \frac{I_f}{\mu}, \quad \hat{I}_m = \frac{I_m}{\mu}.$$

Dropping the hats on the variables, system is equivalent to

$$\begin{cases} S'_f = r_1 S_f + sI_f - u\dfrac{S_f I_m}{N_m} - mS_f N, \\ S'_m = -S_m + \omega I_f - p\dfrac{S_m I_f}{N_f} + \upsilon_1 S_f - mS_m N, \\ I'_f = r_2 I_f + u\dfrac{S_f I_m}{N_m} - mI_f N, \\ I'_m = -qI_m + \upsilon_2 I_f + p\dfrac{S_m I_f}{N_f} - mI_m N, \end{cases} \tag{3}$$

where $r_1 = \frac{\sigma\beta_1 - \mu}{\mu}$, $u = \frac{C\sigma\varsigma}{\mu}$, $r_2 = \frac{\sigma\xi\beta_2 - (\gamma+\mu)}{\mu}$, $s = \frac{\sigma(1-\zeta)\beta_2}{\mu}$, $\omega = \frac{(1-\sigma)(1-\xi)\beta_2}{\mu}$, $u_1 = \frac{(1-\sigma)\beta_1}{\mu}$, $p = \frac{C(1-\sigma)\zeta}{\mu}$, $\upsilon = \frac{(1-\sigma)\zeta\beta_2}{\mu}$, $q = \frac{\mu+\gamma}{\mu}$.

The question about local stability of the system (1) is more conveniently studied by considering the proportions of individuals in the different classes. Consequently we define the variables

$$x = \frac{S_f}{N_m}, \qquad y = \frac{I_f}{N_m}, \qquad z = \frac{I_m}{N_m}$$

where x and y are non-negative, while the variable z requires $0 \le z \le 1$. A simple calculation transforms (3) into the following system of equations

$$\begin{cases} x' = (1+r_1)x + ((q-u-1)z - (\omega+\upsilon_2)y - \upsilon_1 x)x + sy \\ y' = (1+r_2)y + ((q-1)z - (\omega+\upsilon_2)y - \upsilon_1 x)y + uxz \\ z' = (1-q)\left((q-1)z - (\omega+\upsilon_2)y - \upsilon_1 x - p\frac{y}{x+y}\right)z + \left(\upsilon_2 + \frac{p}{x+y}\right)y \end{cases} \tag{4}$$

The complete dynamics of the population can be obtained by solving the system (4) and them using the following differential equation (obtained form system (2))

$$N'_m = \upsilon_1 S_f + (\omega+\upsilon_2)I_f - qI_m - mN_m N$$

written as a Bernoulli equation, we get $N'_m(t) + a(t)N_m(t) = b(t)N_m^2(t)$

where $a(t) = \upsilon_1 x(t) - (1-z(t)) = (\omega+\upsilon_2)y(t) - qz(t)$, $b(t) = -(x(t)+y(t)+1)$

and we obtain $A(T)$ in the following manner: $N'_m(t) = \upsilon^{-1}(t)$

where $\upsilon(t) = rxp\left(\int_0^t a(s)ds\right)\int_0^t -b(s)\exp\left(-\int_0^s a(\sigma)d\sigma\right)ds + \upsilon(0)\exp\left(\int_s^t a(s)ds\right)$

In consequence, the analysis of systems (1), (3) and (5) leads to equivalent results for consideration of nonnegative solutions.

6. Local stability of the disease free equilibrium

Theorem 3. *If* $\tilde{R}_2 < 1 < \tilde{R}_1$ and $\frac{\tilde{R}_2 + \tilde{R}_3}{1-\tilde{R}_2} < \frac{\gamma+\sigma\beta_1}{C\sigma\zeta}$, then the system has a unique disease-free equilibrium (DF) given by the expression

$DF = (\tilde{R}_1 - 1)\left(\frac{\sigma\mu,}{m}, \frac{(1-\sigma)\mu}{m}, 0, 0\right)$ and DF is locally asymptotically stable for system (1).

Proof: The condition $\tilde{R}_1 > 1$ gives us the uniqueness of the (DF) equilibrium of system (1).

Linearizing the system (4) about the corresponding DF equilibrium $(x^*, 0, 0)$ where $x^* = \dfrac{\sigma}{1-\sigma}$, we obtain a linear system whose eigenvalues are

$$\lambda_1 = 1 + r_1 - 2\upsilon_1 x^* = -\frac{\sigma\beta_1}{\mu} < 0$$

and $\lambda_{2,3}$ eigenvalues of the matrix

$$M = \begin{bmatrix} 1 + r_2 - \upsilon_1 x^* & u x^* \\ \dfrac{p}{x^*} + \upsilon_2 & 1 - q - \upsilon_1 x^* \end{bmatrix} \tag{5}$$

Claim: M has negative eigenvalues.
First, we find the trace of (5)

$$Tr(M) = 2(1 - \upsilon_1 x^*) + r_2 - q = 2(1 - \upsilon\tilde{R}_1) + \frac{\mu + \gamma}{\mu}((\tilde{R}_2 - 1) - 1) < 0$$

a simple calculation yields to

$$\det(M) = (1 + r_2 - \upsilon_1 x^*)(1 - q - \upsilon_1 x^*) - (u x^* (\frac{p}{x^*} + \upsilon_2)$$
$$= \left(\frac{\gamma + \sigma\beta_1}{\mu}\right)(\tilde{R}_1 - 1) - \left(\frac{\gamma + \mu}{\mu}\right)\left(\left(\frac{\gamma + \sigma\beta_1}{\mu}\right)(\tilde{R}_2 - 1) + \frac{C\sigma\zeta}{\mu}(\tilde{R}_2 + \tilde{R}_3)\right)$$

under the condition $\dfrac{\tilde{R}_2 + \tilde{R}_3}{1 - \tilde{R}_2} < \dfrac{\gamma + \sigma\beta_1}{C\sigma\zeta}$, we get $\det(M) > 0$. In consequence, M has negative eigenvalues. The claim is proved.

Therefore the DF equilibrium is locally asymptotically stable for the system (4) the theorem was proved.

7. Local stability of the susceptible extinction equilibrium

Theorem 4. *If* $\zeta = 1, \tilde{R}_1 < 1 < \tilde{R}_2$ and $\tilde{R}_2 + \tilde{R}_3 < 1,$ them the system has a unique susceptible extinction equilibrium (SE) given by the expression

$$SE = (\tilde{R}_2 - 1)\left(0, 0, \frac{\sigma(\mu + \gamma)}{m}, \frac{(1 - \sigma)(\mu + \gamma)}{m}\right).$$ and SE is locally asymptotically stable for system (1).

Proof The condition $\tilde{R}_2 > 1$ gives us the uniqueness of the SE equilibrium.

Linearizing the system (4) about the corresponding SE equilibrium $0, y^*, 1)$ where

$$y^* = \frac{\sigma}{1 - \sigma},$$

we obtain a linear system with Jacobian

$$M = \begin{bmatrix} r_1 + q - u - \upsilon_2 y^* & 0 & 0 \\ -\upsilon_1 y^* + u & r_2 + q - 2\upsilon_2 y^* & (q-1)y^* \\ -\upsilon_1 & 0 & q - 1 - \upsilon_2 y^* - p \end{bmatrix} \qquad (6)$$

whose eigen values are

$$\lambda_1 = r_1 + q - u - \upsilon_2 y^* = (\tilde{R}_1 - 1) - \frac{\mu + \gamma}{\mu}(\tilde{R}_2 - 1) - \frac{C\sigma\zeta}{\mu}$$

$$\lambda_2 = r_2 + q - 2\upsilon_2 y^* = -\frac{\sigma\beta_2}{\mu} < 0$$

and $\lambda_3 = q - 1 - p - \upsilon_2 y^* = -\frac{\mu + \gamma}{\mu}(1 - \tilde{R}_2 - \tilde{R}_3)$

under the conditions given by the theorem, $\tilde{R}_1 < 1 < \tilde{R}_2$ and $\tilde{R}_2 + \tilde{R}_3 > 1$, we get $\lambda_{1,3} < 0$. Therefore the SE equilibrium is locally asymptotically stable for the system (4), the theorem was proved.

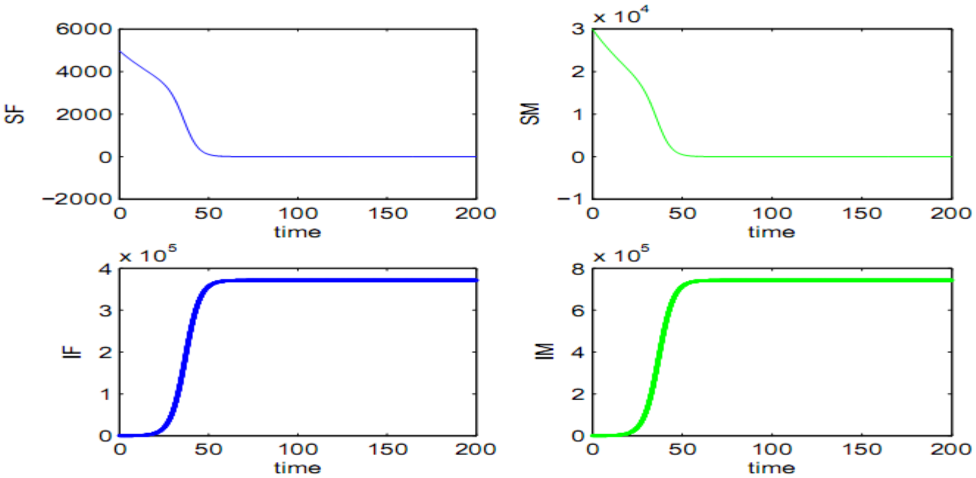

Figure 1: Two sex Epidemic Model. Susceptible Extinction (SE) Equilibrium, with parameter values:

$\beta_1 = 0.0238$; $u_1 = 0.00038$; $\tau = 15$; $C = 0.01935$; $\alpha = 0.1453$; $\beta_2 = 0.003$; $\gamma = 0.072$; $u_2 = u_1 + \gamma$; $k = 41269.113$; $\beta_1 = (\beta_1 * \exp(-u_1 * \tau) - \alpha)/k$; $\eta_1 = 0.02$; $\eta_2 = 0.1453$; $\mu = 0.01453$; $\varsigma = 0.0822$; $\sigma = 1/3$; $\xi = 1$; and Initial conditions; $S_f = 4973$; $S_m = 30000$; $I_f = 50$; $I_m = 27$;

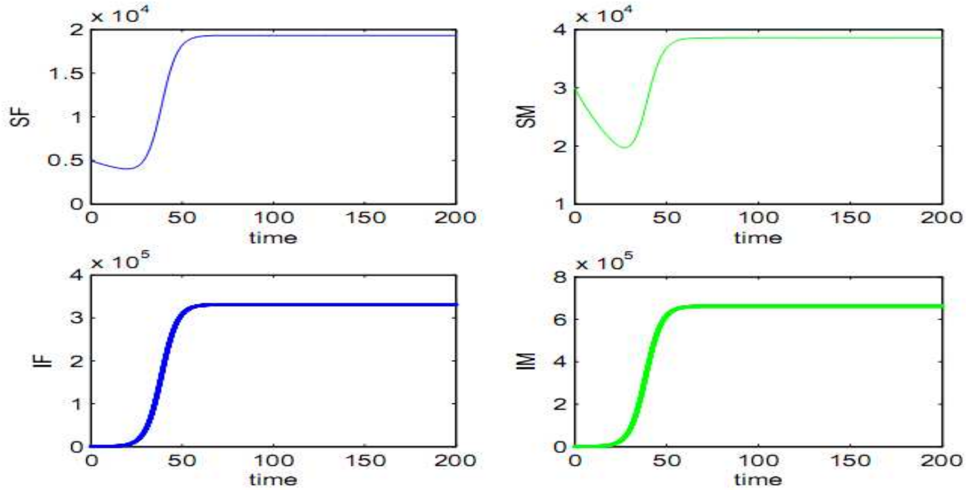

Figure 2: Two sex Epidemic Model. Endemic Equilibrium (EE), with parameter values: $\beta_1 = 0.0238;\ u_1 = 0.00038;\ \tau = 15;\ C = 0.01935;\ \alpha = 0.1453;\ \beta_2 = 0.003;\ \gamma = 0.072;$ $u_2 = u_1 + \gamma;\ k = 41269.113;\ \beta_1 = (\beta_1 * \exp(-u_1 * \tau) - \alpha)/k;\ \eta_1 = 0.02;\ \eta_2 = 0.1453;$ $\mu = 0.01453;\ \varsigma = 0.0822;\ \sigma = 1/3;\ \xi = 0.042;$ and Initial conditions; $S_f = 4973;$ $S_m = 30000;\ I_f = 50;\ I_m = 27;$

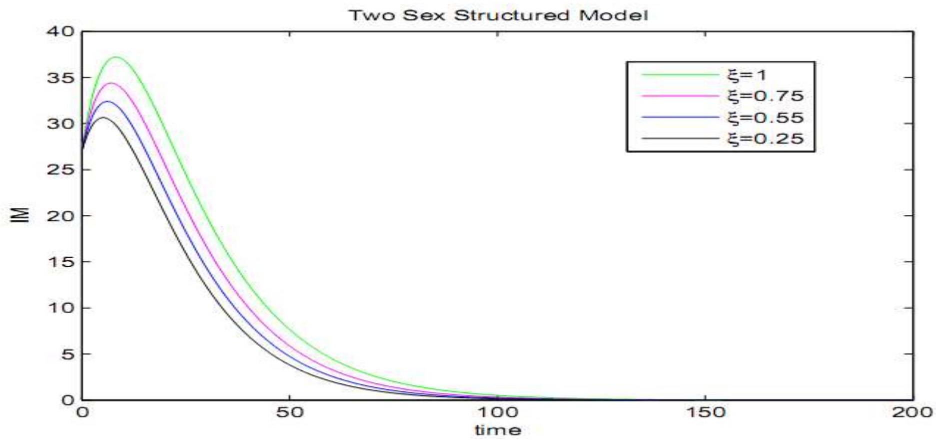

Figure 3: Two sex Epidemic Model Variation of Infected Male Population with respect to the values of portion of Infected Babies (ξ) from Infected Mothers, with parameter values: $\beta_1 = 0.0238;\ u_1 = 0.00038;\ \tau = 15;\ C = 0.01935;\ \alpha = 0.1453;\ \beta_2 = 0.003;$ $\gamma = 0.072;\ u_2 = u_1 + \gamma;\ k = 41269.113;\ \beta_1 = (\beta_1 * \exp(-u_1 * \tau) - \alpha)/k;\ \eta_1 = 0.02;$ $\eta_2 = 0.1453;\ \mu = 0.01453;\ \varsigma = 0.0822;\ \sigma = 1/3;\ \xi = 0.042;$ and Initial conditions; $S_f = 4973;\ S_m = 30000;\ I_f = 50;\ I_m = 27.$

REFERENCE

1. J.Y.T.Mugisha and L.S.Luboobi, Modeling the effect of vertical transmission in the dynamics of HIV/AIDS in an age-structured population, *H. S. Pac. J. Nat Sci.*, 21 (2003) 82-90.
2. J.M.Murray, G.Kaufmann, A.D.Kelleher and D.A.Cooper, A model of primary HIV-1 infection, *Mathematical Biosciences*, 154 (1998) 57-85.
3. J.D.Murray, *Mathematical Biology I. An Introduction*, Springer, 2002, National Aid Trust, Global Statistics, Fact Sheet 3, July 2004.
4. P.Nelson, J.D.Murray and A.S.Perelson, A model of HIV pathogenesis that includes an intracellular delay, *Mathematical Biosciences*, 163 (2000) 201-215.
5. J.Hale, *Ordinary Differential Equations,* Robert E. Krieger Publishing Company, Inc., 1980.
6. N.T.J.Bailey, *The Mathematical Theory of Infectious Diseases*, Grifin, London, 1975.
7. E.Beretta and Y.Kuang, *Modeling and Analysis of a Marine Bacteriophage Infection with Latency Period, Nonlinear Analysis*, Bon Aug.3, 1999.
8. S.P.Blythe and R.M.Anderson, Distributed incubation and in-factious periods in models of the transmission dynamics of the human immunod-eficiency virus (HIV), *IMAJ Math Applied in Medicine and Biology*, 5 (1988) 1-19.
9. P.D.Bohr and P.W.Nelson, Information theory-based selection of mathematical models of hiv infection dynamics, Preprint 2004.
10. F.Brauer, A model for an SI disease in an age-structured population, *Discrete and Continuous Dynamical Systems-Series B*, 2(2) (2002).
11. H.R.Thieme, *Mathematics in Population Biology*, Princeton University Press, Princeton 2003.
12. W.Wang and X.Q.Zhao, An age-structured epidemic model in a patchy environment, *SIAM, J. Appl. Math.*, 65 (2005) 1597-1614.