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Open Distance Pattern Edge Coloring of a Graph

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Abstract. Let $\boldsymbol{G}$ be a connected graph with diameter $\boldsymbol{d}(\boldsymbol{G})$, $\boldsymbol{X} = \{\mathbf{1}, \mathbf{2}, \mathbf{3}, \dots, \boldsymbol{d}(\boldsymbol{G})\}$ be a non-empty set of colors of cardinality $\boldsymbol{d}(\boldsymbol{G})$ and let $\emptyset \neq \boldsymbol{M} \subseteq \boldsymbol{V}(\boldsymbol{G})$. Let $\boldsymbol{f_M^o}$ be an assignment of subsets of $\boldsymbol{X}$ to the vertices of $\boldsymbol{G}$ such that $\boldsymbol{f_M^o}(\boldsymbol{u}) = \{\boldsymbol{d}(\boldsymbol{u}, \boldsymbol{v}) : \boldsymbol{v} \in \boldsymbol{M}, \boldsymbol{u} \neq \boldsymbol{v}\}$ where $\boldsymbol{d}(\boldsymbol{u}, \boldsymbol{v})$ is the distance between $\boldsymbol{u}$ and $\boldsymbol{v}$. We call $\boldsymbol{f_M^o}$ an $\boldsymbol{M}$-Open Distance Pattern Coloring of $\boldsymbol{G}$, an $\boldsymbol{M}$-Open Distance Pattern Edge Coloring of $\boldsymbol{G}$, if no two incident edges have same $\boldsymbol{f}^{\oplus}$, where $\boldsymbol{f_M^{\oplus}} = \boldsymbol{f_M^o}(\boldsymbol{u}) \oplus \boldsymbol{f_M^o}(\boldsymbol{v})$ for every $\boldsymbol{uv} \in \boldsymbol{E}(\boldsymbol{G})$; and if such an $\boldsymbol{M}$ exists then $\boldsymbol{G}$ is called an $\boldsymbol{M}$- Open Distance Pattern Edge Colorable Graph (odpec graph). The minimum cardinality of such an $\boldsymbol{M}$, if it exists, is the$\boldsymbol{M}$-Open Distance Pattern Edge Coloring number of $\boldsymbol{G}$ denoted by $\boldsymbol{\eta_E}(\boldsymbol{G})$.

Keywords: open distance pattern edge coloring, distance pattern labeling of vertices, odpu-graphs, dpd-graphs.

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1. Introduction

For all terminologies which are not defined in this paper, we refer to [9]. All the graphs considered in this paper are finite, simple and connected. Beginning with the origin of the Four Color Problem in 1852, research on graph colorings has developed [4] into one of the most popular areas of graph theory. Historically, graph coloring involved in finding the minimum number of colors to be assigned to the vertices or edges or regions so that adjacent vertices or edges or regions must have different colors.

A proper vertex coloring of a graph G is an assignment of colors to the vertices of G, one color to each vertex, so that adjacent vertices are colored differently. In a k-coloring, we may assume that it is the colors $1,2,\dots,k$ that are being used. While all k colors are typically used in a k-coloring of a graph, there are occasions when only some of the k colors are used.

Interest in edge colorings of graphs was likely inspired by Four Color Problem. The first paper dealing with the edge-coloring problem was written by Tait in 1880 [13]. In

this paper, Tait proved that if the four-color conjecture is true, then the edges of every 3-connected planar graph can be properly colored using only three colors.

Let G be a graph with no loops. [4] *A* k*-edge-coloring* of G is an assignment of k colors to the edges of G in such a way that any two edges meeting at a common vertex are assigned different colors. If G has a k-edge coloring, then G is said to be k-edge colorable. The *chromatic index* of G, denoted by $\chi(G)$, is the smallest k for which G is k-edge-colorable. Let there be given an edge coloring of a graph G of order n.Any two edges of G that are colored the same cannot be adjacent. Hence, one can never have more than $\frac{n}{2}$ edges of G that are of same color.

Coloring of the vertices and edges of a graph G which are required to posses certain conditions have often been motivated by their utility in various applied fields and their intrinsic mathematical interest. An enormous amount of literature has built up on several kinds of colorings of graphs. The classic k-coloring problem tries to assign a color from 1 to k to each vertex in a graph such that no two adjacent vertices share the same color [6]. The k-coloring problem, along with many variations and generalizations, is well studied in both computer science and mathematics. Its applications range from frequency assignment and register allocation, to circuit board testing and timetable scheduling [See 3, 5, 14].

On 29th November 2006, B.D. Acharya [7] conveyed to the second author the following definitions for a detailed study.

Let $G = (V, E)$ be a given connected simple (p, q) -graph with diameter $d_G, \emptyset \neq M \subseteq V(G)$ and $u \in V(G)$. Then, the *M-distance-pattern* of u is the set $f_M(u) = \{d(u, v): v \in M\}$. Clearly, $f_M(u) = \{j: N_j^M[u] \neq \emptyset\}$, where $N_j^M[u] = \{v \in M: d(u, v) = j\}$. If $f_M: u \mapsto f_M(u)$ is an injective function, then the set M is a *distance-pattern distinguishing set* (or, a 'dpd-set' in short) of G. A graph G with a dpd-set is called a*distance-pattern distinguishing graph* (*dpd-graph*) [7]. We associate with each vertex u of a graph $G = (V, E)$ its *openM-distance pattern* (or, 'odp' in short), $f_M^0(u) = \{d(u, v): v \in M, u \neq v\}$, and the graphs in which every vertex has the same open distance pattern are called *odp-uniform graphs* (or, simply, 'odpu-graphs'), where the *set-valued function* (or, *set-valuation*) [7] f_M^o is called the *open distance pattern uniform* (or, a *odpu*)- *function* and M is called an *odpu-set* of G. The minimum cardinality of a dpd-set (odpu-set) in G, if it exists, is the *dpd-number* (odpu-number) of G.

Motivated from the definition of dpd (odpu)-graphs and the classic k-coloring problem, we defined M-open distance pattern coloring of a graph Gin [8] as follows.

Definition 1. [8] Let G be a connected graph with diameter $d(G, X) = \{1, 2, 3, \ldots, d(G)\}$ be a non-empty set of colors of cardinality $d(G)$ and let $\emptyset \neq M \subseteq V(G)$. Let f_M^o be an assignment of subsets of X to the vertices of G such that $f_M^o(u) = \{d(u, v): v \in M, u \neq v\}$ where $d(u, v)$ is the distance between u and v. Given such a function f_M^o for all vertices in G, an induced edge function $f_M^{\oplus}$ of an edge $uv \in E(G)$, $f_M^{\oplus}(uv) = f_M^o(u) \oplus f_M^o(v)$. We call f_M^o an M-open distance pattern coloring of G, if no two adjacent vertices have same f_M^o and if such an M exists for a graph G, then G is called an open distance pattern colorable graph(odpc-graph); the minimum cardinality of such an M if it exists, is the open distance pattern coloring number of G, denoted by $\eta_M(G)$.

In [8], we proved that the stars; trees; C_n; $n \neq 3, 5, 7$; unicycle graphs G with $G \ncong C_n$, $n = 3, 5, 7$, complete bipartite graphs; are some classes of M-open distance pattern colorable graphs.

In this paper, as an analogue of an M-open distance pattern colorable graph, we define M −Open Distance Pattern Edge Coloring of a graph as follows.

Definition 2. We call f_M^o an *M-Open Distance Pattern Coloring* of G, an M-Open Distance Pattern edge Coloring of G, if no two incident edges have same $f^{\oplus}$, where $f_M^{\oplus} = f_M^o(u) \oplus f_M^o(v)$ for every $uv \in E(G)$; and if such an M exists then G is called an M- Open Distance Pattern Edge Colorable Graph (odpec graph). The minimum cardinality of such an M, if it exists, is the M-Open Distance Pattern Edge Coloring number of G denoted by $\eta_E(G)$.

Complete graphs $K_n, \forall n \geq 1$, C_n,; $n = 3, 5, 7$ are not M-odpec.

An example of M-open distance pattern edge colorable graph is path P_5. For, let $V(P_5)$ be $\{v_1, v_2, v_3, v_4, v_5\}$. Then $M = \{v_1, v_2, v_5\}$ is an M-odpec set since the incident edges receive non-identical $f_M^{\oplus}$.

However, not all M-odpc graphs are M-Open Distance Pattern Edge Colorable. For example consider $K_{1,n}$. $K_{1,n}$ is M-odpc when M is chosen as any two vertices in the spokes of $K_{1,n}$ and this is the only choice of M. for $K1, n$ to be odpec. With this choice of M, no two adjacent vertices have same f_M^o whereas $f_M^{\oplus}$ of all the edges receives the labeling namely $\{1,2\}$. Hence $K_{1,n}$ is M-odpc, but not M-odpec.

Lemma 3. M-open distance pattern edge coloring number of a graph G, is greater than 2.
Proof: We have $|M| \geq 2$ for an M-odpc graph [8]. Hence it is enough to show that $|M| \neq 2$. If possible, $|M| = 2$. say $M = \{u, v\}$.
Case 1: $uv \in E(G)$. This is not possible since $f_M^o(u) = f_M^0(v) = \{1\}$.
Case 2: $uv \notin E(G)$. There exists a $u - v$ path say $u = v_1 v_2 \ldots v_n = v$ of length $n - 1$.
Subcase 1: n is even. Then, for the two middle most vertices $v_{\frac{n}{2}}, v_{\frac{n}{2}+1}$ of the $u - v$ path

$$f_M^o(v_{\frac{n}{2}}) = f_M^o(v_{\frac{n}{2}+1}) = \{\frac{n}{2}, \frac{n}{2} - 1\}.$$

Subcase 2: n is odd.
$f_M^o(v_{\lceil\frac{n}{2}\rceil}) = f_M^o(v_{\lceil\frac{n}{2}\rceil+1}) = \{\lceil\frac{n}{2}\rceil, \lceil\frac{n}{2}\rceil - 2\}$. $f_M^{\oplus}$ $(v_{\lceil\frac{n}{2}\rceil-1} v_{\lceil\frac{n}{2}\rceil}) = f_M^{\oplus}$ $(v_{\lceil\frac{n}{2}\rceil} v_{\lceil\frac{n}{2}\rceil+1}) = \{\lceil\frac{n}{2}\rceil - 2, \lceil\frac{n}{2}\rceil - 1, \lceil\frac{n}{2}\rceil\}$.□

It is interesting to note that odpc number of a graph is always less than or equal to odpec number. Also odpc number of any graph is always greater than 2. Trees, complete bipartite graphs, even cycles have odpc number 2.

Proposition 4. An open distance pattern colorable graph G is Open Distance Pattern Edge Colorable if and only if there exists no vertex $v_i \in V(G)$ such that $f_M^{\oplus}(v_{i+j}) = f_M^{\oplus}(v_{i+k})$; $j, k = 1,2, \ldots, d(v_i)$, where M, is the odpec set of G.

Proof: Let G be the odpc graph with odpc set M. Let $f_M^o(v_{i+j}) = f_M^o(v_{i+k})$ for some j, k.

Then $f_M^o(v_i) \oplus f_M^o(v_{i+j}) = f_M^o(v_i) \oplus f_M^o(v_{i+k})$, which implies $f_M^{\oplus}(v_iv_j) = f_M^{\oplus}(v_iv_k)$ a contradiction to the definition of $f_M^{\oplus}$.

Conversely, let for no vertex $v_i \in V(G)$ there exists v_{i+j}, $v_{i+k} \in V(G)$ such that, $f_M^o(v_{i+j}) = f_M^o(v_{i+k})$. That is, for every $j, k = 1,2,\ldots, d(v_i), f_M^o(v_{i+j}) \neq f_M^o(v_{i+k})$. This in turn implies that for every $f_M^{\oplus}(v_iv_j) \neq f_M^{\oplus}(v_iv_k)$. That is, no two incident edges have the same open distance pattern, and hence G is an odpec graph. □

Some Classes of M −odpec graphs

Theorem 5. Path P_n is Open Distance Pattern Edge Colorable if and only if $n > 4$.

Proof: Let G be a graph isomorphic to P_n, $n > 4$. Let $v_1, v_2, \ldots, v_n$ be the vertices of P_n. In [8] we proved that P_n is M-Open Distance Pattern Colorable.

Choose $M = \{v_1, v_2, v_n\}$.

$f_M^o(v_1) = \{1, d(G)\}$; $f_M^o(v_2) = \{1, d(G) - 1\}$; $f_M^{\oplus}(v_1v_2) = \{d(G), d(G) - 1)\}$.

$f_M^o(v_i) = \{i - 2, i - 1, d(G) - (i - 1)\}$; for $i = 3,4, \ldots, n$.

$f_M^{\oplus}(v_iv_{i+1}) = \{i, i - 2, d(G) - i, d(G) - (i - 1)\}$; for $i = 2, 3,4, \ldots, n - 1$.

One may easily verify that no two adjacent vertices have same f_M^o and no two incident edges have same $f_M^{\oplus}$ .So the path P_n, $n > 4$, is an open distance pattern edge colorable graph.

Conversely, assume $n \leq 4$.

Case 1: $n = 4$

Let P_4: $v_1v_2v_3v_4$ be path with some M −odpec set say, M. By Lemma 3 $|M| > 2$. Hence the different possibilities for the choices of M are

Subcase 1: $|M| = 3$

$M_1 = \{v_1, v_2, v_3\} f_M^o(v_1) = f_M^o(v_3) = \{1, 2\}$, $f_M^o(v_2) = \{1\}$, $f_M^{\oplus}(v_1v_2) = f_M^{\oplus}(v_2v_3) = \{2\}$which is not possible. Similar is the case when $M_2 = \{v_1, v_2, v_4\}$

Subcase 2: $|M| = 4$

$M = \{v_1, v_2, v_3, v_4\}$. Then $f_M^o(v_2) = f_M^o(v_3) = \{1, 2\}$, a contradiction.

Case 2: $n = 3$.

The only possibility is $M = \{v_1, v_2, v_3\}$, since by Lemma 3 $|M| > 2$. When $M = \{v_1, v_2, v_3\}, f_M^o(v_1) = f_M^o(v_3) = \{1, 2\}$, $f_M^o(v_2) = \{1\}$, $f_M^{\oplus}(v_1v_2) = f_M^{\oplus}(v_2v_3) = \{2\}$. A contradiction. Hence P_n; $n \leq 4$ is not M-odpec. □

[2] Given a graph G we denote by G^+ the graph obtained from G by augmenting a new vertex v' for each vertex v of G and augmenting a new edge vv'. In particular, P_n^+ is called a comb.

Theorem 6. Comb $G \cong P_n^+$ is M-odpec if and only if diameter of P_n^+ is greater than 3.

Proof: Consider the comb $G \cong P_n^+$ of $2n$ vertices; with diameter is greater than 3. Let $v_1, v_2, v_3, \dots, v_n$ be the vertices of its stem P_n and $u_1, u_2, u_3, \dots, u_n$ the pendant vertices. Choose $M = \{v_1, u_1, u_n\}$. $f_M^o(u_1) = \{1,\ d(G)\}$. $f_M^o(v_1) = \{1,\ d(G) - 1\}$. $f_M^o(v_i) = \{i - 1,\ i,\ d(G) - i\}$. $f_M^o(u_i) = \{i,\ i + 1,\ d(G) - (i - 1)\}$; $i = 1, 2, \dots, n$. $f_M^{\oplus}\ (v_1u_1) = \{d(G),\ d(G) - 1\}$. $f_M^{\oplus}\ (v_iv_{i+1}) = \{i - 1,\ i + 1,\ d(G) - i,\ d(G) - (i + 1)\}$; $i = 1, 2, \dots, n - 1$. $f_M^{\oplus}\ (v_iu_i) = \{i - 1,\ i + 1,\ d(G) - i,\ d(G) - (i - 1)\}$; $i = 2,3, \dots, n$. Therefore, $f_M^{\oplus}$ is distinct for incident edges.

Conversely, assume comb is odpec for diameter less than 4.Only possibility is comb with diameter 3, since cardinality of an odpec-set is greater than 2. Let v_1, v_2 be the vertices of its stem and u_1, u_2 be the pendant vertices. Then P_n^+ is isomorphic to path P_4,which is not odpec by Theorem5. Hence, the comb with diameter greater than 3 is an open distance pattern edge colorable graph with $\eta_E(G) = 3$ since $\eta_E(G) > 2$ by Lemma 3. □

[10] *An olive tree* is a rooted tree consisting of n branches, where the ith branch is a path of length i.

Theorem 7. Olive tree T is open distance pattern edge colorable if and only if $n > 2$, where n is the number of branches of the olive tree.

Proof: Assume $n > 2$.
Let the Olive tree T be formed by joining P_i, $i = 1, 2, \dots, n$ paths joined at v, where P_i has length i. Olive tree T have vertices $v_{i0}, v_{i1}, v_{i2}, \dots, v_{ii}$ where $1 \le i \le n$, and v_{i0} is identified as v. Let $M = \{v_{ii}\}$, $1 \le i \le n$ the pendant vertices of $P_i{}'si = 1,2, \dots, n$, of the olive tree. $f_M^o(v) = \{1, 2, \dots, n\}$ and $f_M^o(v_{i1})$ do not contain $i + 1$ for $i = 1, 2, \dots, n$. Hence all $f_M^{\oplus}\ (vv_{i1})$ is distinct for $i = 1,2, \dots, n$. $f_M^o(v_{ij}) = \{j + 1, j + 2, \dots, j + n - 1, i - j\}$; $1 \le i \le n$; $1 \le j \le i$. For any vertex in the olive tree ($n > 2$), all the vertices adjacent to the vertex v_{ij} have distinct f_M^o. Therefore, $f_M^{\oplus}$ is distinct for all incident edges.

Conversely, assume $n \le 2$. When $n = 2$, $T \cong P_4$; and when $n = 1$; $T \cong K_2$ which are not odpec. □

Theorem 8. The cycle C_n, is an open distance pattern edge colorable if and only if $n \ge 8$.
Proof: Assume cycle C_n is open distance pattern edge colorable. Let $V(C_n) = \{v_1, v_2, \dots, v_n\}$. Choose $M = \{v_1, v_2, v_5\}$.

Case 1: n is odd. Let $n \ge 9$.
Choose $M = \{v_1, v_2, v_5\}$.
$f_M^o(v_1) = \{1, 4\}$; $f_M^o(v_2) = \{1, 3\}$; $f_M^o(v_3) = \{1, 2\}$; $f_M^o(v_4) = \{1, 2, 3\}$,
$f_M^o(v_i) = \{i - 1, i - 2, i - 5\}$; for $i = 5, 6, \dots, \lceil\frac{n}{2}\rceil$,
$f_M^o(v_i) = \{n - (i - 1), (i - 2), i - 5\}$; for $i = \lceil\frac{n}{2}\rceil + 1$,

$f_M^o(v_i) = \{n-(i-1), n-(i-2), i-5\}$;for$\lceil\frac{n}{2}\rceil + 2 \le i \le \lceil\frac{n}{2}\rceil + 4$,
$f_M^o(v_i) = \{n-(i-1), n-(i-2), n-(i-5)\}$;for$\lceil\frac{n}{2}\rceil + 5 \le i \le n$,
$f_M^{\oplus}(v_i v_{i+1}) = \{i,\ i-2,\ i-5,\ i-4)\}$, for$i = 5, 6, \dots, \lceil\frac{n}{2}\rceil - 1$,
$f_M^{\oplus}(v_{\lceil\frac{n}{2}\rceil} v_{\lceil\frac{n}{2}\rceil+1}) = \{\frac{n-3}{2}, \frac{n-7}{2}, \frac{n-9}{2}\}$; $f_M^{\oplus}(v_{\lceil\frac{n}{2}\rceil+4} v_{\lceil\frac{n}{2}\rceil+5}) = \{\frac{n-9}{2}, \frac{n-5}{2}\}$
$f_M^{\oplus}(v_i v_{i+1}) = \{n-(i-1),\ n-(i-2),\ n-(i-5),\ n-(i-4)\}$, for $i = \lceil\frac{n}{2}\rceil + 5, \dots, n-1$,
$f_M^{\oplus}(v_n v_1) = \{2,\ 4,\ 5,\ \}$. No two incident edges have same $f_M^{\oplus}$.

Case 2: n is even and ≥ 8.

$$f_M^o(v_1) = \{1,4\};\ f_M^o(v_2) = \{1,3\};\ f_M^o(v_3) = \{1,2\};\ f_M^o(v_4) = \{1,2,3\};$$
$$f_M^o(v_i) = \{i-1, i-2, i-5\};\ 5 \le i \le \frac{n}{2} + 1,$$
$$f_M^o(v_i) = \{n-(i-1), n-(i-2), i-5\};\ \frac{n}{2} + 1 < i \le \frac{n}{2} + 4,$$
$$f_M^o(v_i) = \{n-(i-1), n-(i-2), n-(i-5)\};\ \frac{n}{2} + 4 < i \le n,$$

$f_M^{\oplus}(v_1 v_2) = \{3,4\}$; $f_M^{\oplus}(v_2 v_3) = \{2,3\}$; $f_M^{\oplus}(v_3 v_4) = \{3\}$; $f_M^{\oplus}(v_4 v_5) = \{1,2,4\}$,
$f_M^{\oplus}(v_i v_{i+1}) = \{i,\ i-2,\ i-4,\ i-5\}$, for$i = 5,6, \dots, \frac{n}{2}$,
$f_M^{\oplus}(v_i v_{i+1}) = \{\frac{n}{2} - 4,\ \frac{n}{2} - 3\}$,for$i = \frac{n}{2} + 1$,
$f_M^{\oplus}(v_i v_{i+1}) = \{n-i, n-(i-2),\ (i-4),\ (i-5)\}$,for$\frac{n}{2} + 2 \le i \le \frac{n}{2} + 3$,
$f_M^{\oplus}(v_i v_{i+1}) = \{\frac{n}{2} - 4,\ \frac{n}{2} - 2,\ \frac{n}{2} - 1,\ \frac{n}{2}\}$,for$i = \frac{n}{2} + 4$,
$f_M^{\oplus}(v_i v_{i+1}) = \{n-i,\ n-(i-2),\ n-(i-4),\ n-(i-5)\}$, for $i = \frac{n}{2} + 5, \dots, n-1$.

$$f_M^{\oplus}(v_1 v_n) = \{2,\ 4,\ 5\},$$

Hence, no two incident edges have identical $f_M^{\oplus}$, and hence, cycle $C_n, n \ge 8$ is an open distance pattern edge colorable.

Conversely, assume $n < 8$ is M-odpec. In [8] we proved that C_n; $n = 3, 5, 7$ is not odpc and hence not odpec. Hence, we consider C_4 and C_6.Let C_4: $v_1 v_2 v_3 v_4$ be a cycle with open distance pattern edge colorable set $M. |M| \ge 3$. The different possibilities for the choice of M are

Case 1: $|M| = 3$.
$M = \{v_1, v_2, v_3\}$ (or $\{v_2, v_3, v_4\}$ or $\{v_3, v_4, v_1\}$ or $\{v_4, v_1, v_2\}$.) Then, $f_M^o(v_1) = f_M^o(v_3) = \{2\}$, $f_M^o(v_2) = \{1\}$. But $f_M^{\oplus}(v_1 v_2) = f_M^{\oplus}(v_2 v_3) = \{2\}$.

Case 2: $|M| = 4$.
Then, $M = \{v_1, v_2, v_3, v_4\}$ and $f_M^o(v_1) = f_M^o(v_2) = f_M^o(v_3) = f_M^o(v_4) = \{1,2\}$.
Consider C_6: $v_1 v_2 v_3 v_4 v_5 v_6$ be a cycle with open distance pattern edge colorable set $M. |M| \ge 3$. The different possibilities for the choice of M are

Case 1:$|M| = 3$.
Choose $M_1 = \{v_1, v_2, v_3\}$ (or $\{v_2, v_3, v_4\}$ or $\{v_3, v_4, v_5\}$ or $\{v_4, v_5, v_6\}$ or $\{v_5, v_6, v_1\}$or $\{v_6, v_1, v_2\}$). Then

$$f^o_{M_1}(v_1) = f^o_{M_1}(v_3) = \{1,2\},\ f^o_{M_1}(v_2) = \{1\};\ f^{\oplus}_{M_1}\ (v_1v_2) = f^{\oplus}_{M_1}\ (v_2v_3) = \{2\}.$$

Next, choose $M_2 = \{v_1, v_2, v_4\}$. or($\{v_2, v_3, v_5\}$ or $\{v_3, v_4, v_6\}$ or $\{v_4, v_5, v_1\}$ or $\{v_5, v_6, v_2\}$ or $\{v_6, v_1, v_3\}$). Then,
$f^o_{M_2}(v_2) = f^o_{M_2}(v_3) = \{1,2\}, M_3 = \{v_1, v_3, v_5\}$or ($\{v_2, v_4, v_6\}$).

$$f^o_{M_3}(v_1) = f^o_{M_3}(v_3) = \{2\},\ f^o_{M_3}(v_2) = \{1,3\}.\ f^{\oplus}_{M_3}(v_1v_2) = f^{\oplus}_{M_3}(v_2v_3) = \{1,2,3\}.$$

Case 2:$|M| = 4$.
Choose $M_1 = \{v_1, v_2, v_3, v_4\}$(or$\{v_2, v_3, v_4, v_5\}$ or $\{v_3, v_4, v_5, v_6\}$ or $\{v_4, v_5, v_6, v_1\}$ or $\{v_5, v_6, v_1, v_2\}$ or $\{v_6, v_1, v_2, v_3\}$). Then, $f^o_{M_1}(v_2) = f^o_{M_1}(v_3) = \{1,2\}$.
Again, choose $M_2 = \{v_1, v_2, v_3, v_5\}$ (or $\{v_2, v_3, v_4, v_6\}$ or $\{v_3, v_4, v_5, v_1\}$ or $\{v_4, v_5, v_6, v_2\}$ or $\{v_5, v_6, v_1, v_3\}$ or $\{v_6, v_1, v_2, v_4\}$.) Then, $f^o_{M_2}(v_1) = f^o_{M_2}(v_3) = \{1,2\}$; $f^o_{M_2}(v_2) = \{1,3\}$ $f^{\oplus}_{M_2}\ (v_1v_2) = f^{\oplus}_{M_2}\ (v_2v_3) = \{2,3\}$.
Now, choose $M_3 = \{v_1, v_2, v_4, v_5\}$(or$\{v_2, v_3, v_5, v_6\}$ or $\{v_3, v_4, v_6, v_1\}$). Then $f^o_{M_3}(v_1) = f^o_{M_3}(v_2) = \{1,2,3\}$.

Case 3:$|M| = 5$.
Choose $M = \{v_1, v_2, v_3, v_4, v_5\}$ (or $\{v_2, v_3, v_4, v_5, v_6\}$ or $\{v_3, v_4, v_5, v_6, v_1\}$ or $\{v_4, v_5, v_6, v_1, v_2\}$ or $\{v_5, v_6, v_1, v_2, v_3\}$ or $\{v_6, v_1, v_2, v_3, v_4\}$.)Then, $f^o_M(v_1) = f^o_M(v_2) = \{1,2,3\}$.
Case 4:$|M| = 6$.
Choose $M = \{v_1, v_2, v_3, v_4, v_5, v_6\}$ so that $f^o_M(v_1) = f^o_M(v_2) = \{1,2,3\}$. Hence, no choice of M admit an M-odpec. Hence, $C_n;\ n < 8$ is not M-odpec. □

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