Annals of Pure and Applied Mathematics
Vol. 6, No. 2, 2014, 170-177
ISSN: 2279-087X (P), 2279-0888(online)
Published on 21 July 2014
www.researchmathsci.org

Annals of
Pure and Applied Mathematics

Applications of Riemannian Geometry Comparing with Symplectic Geometry

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Received 24 June 2014; accepted 30 June 2014

Abstract. The main object of this paper is to provide various applications of Riemannian geometry in the theory of relativity and to provide special comparisons between symplectic geometry and Riemannian geometry with respect to different point of view.

Keywords: Riemannian geometry, relativity, symplectic geometry

AMS Mathematics Subject Classification (2010): 70G45

1. Introduction

Riemannian geometry is the branch of differential geometry [9] that studies Riemannian manifolds, smooth manifolds with a Riemannian metric. Riemannian geometry was first put forward in generality by *Bernhard Riemann* in the nineteenth century. One of the main notions of the Riemannian geometry is the notion of connection [3], [10]. The connection (or parallel transport) allows to compare what is happening at two distant points of a curved space, in spite of the fact that there is no direct and immediate way to communicate between these points. Earlier, in the 1910's, *Albert Einstein* discovered that the Riemannian geometry can be successfully used to describe general relativity theory which is in fact a classical theory of gravitation. By its intrinsic beauty, as well as by wealth of applications the Riemannian geometry lies at the core of modern mathematics.

2. Einstein's Field Equations (EFE)

According to Einstein, matter is the cause of the gravitational field and the causative matter is described in his theory by a mathematical object called the energy-momentum tensor, which is coupled to geometry (i.e. space time) by his field equations, so that matter causes space-time curvature (his gravitational field) and space-time constrains motion of matter when gravity alone acts. According to the astrophysics community [8], Einstein's field equations,
"... couple the gravitational field (contained in the curvature of space time) with its sources." (Foster & Nightingale 1995).

"*Again, just as the electric field, for its part, depends upon the charges and is instrumental in producing mechanical interaction between the charges, so we must assume here that the metrical field (or, in mathematical language, the tensor with components g_{ik}) is related to the material filling the world.*" (Weyl 1952).

"*...we have, in following the ideas set out just above, to discover the invariant law of gravitation, according to which matter determines the components $\Gamma^{\alpha}_{\beta i}$ of the gravitational field, and which replaces the Newtonian law of attraction in Einstein's Theory.*" (Weyl 1952).

"*Thus the equations of the gravitational field also contain the equations for the matter (material particles and electromagnetic fields) which produces this field.*" (Landau & Lifshitz 1951).

"*Clearly, the mass density, or equivalently, energy density $\rho(\vec{x},t)$must play the role as a source. However, it is the 00 component of a tensor $T_{\mu\nu}(x)$, the mass-energy-momentum distribution of matter. So, this tensor must act as the source of the gravitational field.*" (Hooft 2009).

"*In general relativity, the stress-energy or energy-momentum tensor T^{ab} acts as the source of the gravitational field. It is related to the Einstein tensor and hence to the curvature of space time via the Einstein equation.*" (McMahon 2006).

The space-time geometry is described by a mathematical object called Einstein's tensor, $G_{\mu\upsilon}$, ($\mu,\upsilon=$ 0, 1, 2, 3). Einstein's field equations are therefore

$$G_{\mu\upsilon}=R_{\mu\upsilon}-\frac{1}{2}g_{\mu\upsilon}R=-kT_{\mu\upsilon}$$

$R_{\mu\upsilon}$ is called the Ricci tensor and R the Ricci curvature. If $T_{\mu\upsilon}=0$ then one finds that $R=0$ and this expression according to Einstein allegedly reduces to

$$R_{\mu\upsilon}=0$$

and is said to describe a universe that contains no matter (the so-called static empty universe). In Riemannian geometry, the Ricci curvature tensor $R_{\mu\upsilon}$ must be symmetric in μ and υ, i.e. $R_{\mu\upsilon}=R_{\upsilon\mu}$.

Therefore the Einstein curvature tensor $G_{\mu\upsilon}$ is a symmetric second-rank tensor that is a function of the metric. The Einstein tensor is of crucial physical significance in general theory of relativity [6], since it can be shown from the Bianchi identities that,

$$G_{\mu\upsilon;\upsilon}=0$$

In general theory of relativity, the Einstein curvature tensor models local gravitational forces and it is equal (up to a gravitational constant) to the stress-energy tensor

$$G_{\mu\upsilon;\upsilon}=T_{\mu\upsilon;\upsilon}$$

Einstein took the solution of these equations to be of the form

$$G_{\mu\upsilon}+g_{\mu\upsilon}\wedge=kT_{\mu\upsilon}$$

where we can determine the constant k by requiring that we should recover the laws of Newtonian gravity and dynamics in the limit of a weak gravitational field and non-relativistic motion. In fact k turns out to equal$\frac{8\pi G}{c^4}$.

Using geometrized units where$G = c = 1$, this can be rewritten as

$$G_{\mu\upsilon} + g_{\mu\upsilon} \wedge = 8\pi T_{\mu\upsilon}$$

The expression on the left represents the curvature of space time as determined by the metric; the expression on the right represents the matter/energy content of space-time. The EFE can then be interpreted as a set of equations dictating how matter/energy determines the curvature of space time.

These equations, together with the geodesic equation, which dictates how freely-falling matter moves through space-time, form the core of the mathematical formulation of general theory of relativity [7].

Theorem 2.1. Vacuum solutions of Einstein's equation are Einstein manifolds.

Proof: In local coordinates the condition that (M, g) be an Einstein manifold is

$$R_{\mu\upsilon} = \mu \, g_{\mu\upsilon} \tag{2.1}$$

Taking the trace of both sides of (2.1) we get,

$$R = n \, \mu$$

In general theory of relativity, Einstein's equation with a cosmological constant $\wedge$ is

$$R_{\mu\upsilon} - \frac{1}{2} g_{\mu\upsilon} R + g_{\mu\upsilon} \wedge = 8\pi T_{\mu\upsilon}$$

In a vacuum, $T_{\mu\upsilon} = 0$ and one can rewrite Einstein's equation in the form (assuming n>2):

$$R_{\mu\upsilon} = \frac{2\wedge}{\text{n}-2} \, g_{\mu\upsilon} \tag{2.2}$$

Comparing (2.1) and (2.2), we can say that, vacuum solutions of Einstein's equation are Einstein manifolds with μ proportional to the cosmological constant. □

Theorem 2.2. Einstein universe is not an Einstein space.

Proof: An Einstein space is characterized by the property

$$R_{ij} = \frac{R}{n} g_{ij} \tag{2.3}$$

To examine Einstein universe, Einstein line element is given by

$$ds^2 = -\left(1 - \frac{r^2}{R^2}\right)^{-1} dr^2 - r^2(d\theta^2 + sin^2\theta d\varphi^2) + dt^2$$

Here we get,

$$R_{\mu\mu} = \frac{2}{R^2} \, g_{\mu\mu} \quad \text{where, } \mu = 1, 2, 3 \tag{2.4}$$

Also, $R_{44} = 0$, $R_{\mu\upsilon} = 0$ for $\mu \neq \upsilon$

$$R = g^{\mu\upsilon} R_{\mu\upsilon} = \sum_{\mu=1}^{4} g^{\mu\mu} R_{\mu\mu} = \sum_{\mu=1}^{4} \frac{R_{\mu\mu}}{g_{\mu\mu}}$$

$$= \frac{2}{R^2} (1+ 1+ 1 + 0) \qquad \text{[by (2.4)]}$$

That is, $\frac{R}{3} = \frac{2}{R^2}$

Using this in (2.4), we get

$$R_{\mu\mu} = \frac{R}{3}\, g_{\mu\mu}$$

Also, $R_{44} = 0$, $g_{44} \neq 0$ and $R_{\mu\upsilon} = 0$, for $\mu \neq \upsilon$.
These facts prove that,

$$R_{\mu\upsilon} \neq \frac{R}{4}\, g_{\mu\upsilon}$$

This implies that, Einstein universe is not an Einstein space. □

3. Geodesics

According to Newton's laws the 'natural' trajectory of a particle which is not being acted upon by any external force is a straight line. In general theory of relativity, since gravity manifests itself as spacetime curvature, these 'natural' straight line trajectories generalize to curved paths known as *geodesics*. These are defined physically as the trajectories followed by freely falling particles, i.e., particles which are not being acted upon by any non-gravitational external force. Geodesics [1] are defined mathematically as spacetime curves that parallel transport their own tangent vectors. For metric spaces i.e. spaces on which a metric function can be defined, we can also define geodesics as external paths in the sense that along the geodesic between two events E_1 and E_2, the elapsed proper time is an extremum, i.e.

$$\delta \int_{E_1}^{E_2} d\tau = 0$$

Mathematically, the curvature of spacetime can be revealed by considering the deviation of neighbouring geodesics [2].

The worldline of a material particle may be written with the proper time, τ, as parameter along the worldline. The four velocity of the particle is the tangent vector to the worldline. The geodesic equation for the particle is

$$\frac{d^2x^\mu}{d\tau^2} + \Gamma^\mu_{\upsilon\lambda}\frac{dx^\upsilon}{d\tau}\frac{dx^\lambda}{d\tau} = 0$$

4. Symplectic Geometry versus Riemannian Geometry

Symplectic geometry is the geometry of a closed skew-symmetric form. It turns out to be very different from the Riemannian geometry with which we are familiar. One important difference is that, in some intrinsic way they do not involve derivatives. Thus, symplectic geometry is essentially topological in nature. Another important feature is that, it is a 2-dimensional geometry that measures the area of complex curves instead of the length of real curves. On the other hand, a Riemannian manifold, or Riemann space, is a smooth manifold whose tangent spaces are endowed with inner product with satisfying some conditions. Euclidean spaces are also Riemann spaces. Smooth surfaces in Euclidean spaces are Riemann spaces. A hyperbolic non-Euclidean space is also a Riemann space. A curve in a Riemann space has the length. A Riemann space is both a smooth manifold and a metric space; the length of the shortest curve is the distance. The angle between two curves intersecting at a point is the angle between their tangent lines. Also, the study of Riemannian manifolds is called Riemannian geometry.

Definition 4.1. A symplectic form ω on a smooth manifold M is a smooth 2-form ω on M that is closed and non-degenerate, so $\omega \in \Omega^2(M)$ with $d\omega = 0$ and ω_x is non-degenerate on T_xM for all $x \in M$.

A necessary condition for the existence of a symplectic form ω on M is that M should have even dimension $2n$. Moreover $\omega^n/n!$ is a volume form (the so called Liouville form) giving M an orientation. If in addition M is connected and compact then the even dimensional De Rham cohomology spaces $H^{2p}_{DR}(M)$ should be all non-zero for $0 \le p \le n$. Indeed $[\omega]^n = [\omega^n] \neq 0$ which in turn implies that $[\omega^p] = [\omega]^p \neq 0$ for $0 \le p \le n$.

Definition 4.2. A symplectic vector space is a pair (E, ω) consisting of a finite-dimensional real vector space E and a non-degenerate, skew-symmetric bilinear form ω: $E \times E \to \mathbb{R}$.

Definition 4.3. Let (E, ω) be a symplectic vector space and $F \subset E$ be a subspace. The symplectic complement of F is the subspace $F^{\perp} = \{X \in E \mid \omega(X, Y) = 0, \forall\, Y \in F\}$.
The properties of the symplectic complement are given by follows:
Let F and G be the subspaces of a symplectic vector space (E, ω) then,

(i) If $F \subset G$ then $F^{\perp} \supset G^{\perp}$ (ii) $(F^{\perp})^{\perp} = F$ (iii) $(F + G)^{\perp} = F^{\perp} \cap G^{\perp}$
(iv) $(F \cap G)^{\perp} = F^{\perp} + G^{\perp}$ (v) $dim\, F^{\perp} = dim\, E - dim\, F$

Definition 4.4. A subspace $F \subset E$ of a symplectic vector space is called
(a) Isotropic if $F \subset F^{\perp}$ (b) Co-isotropic if $F^{\perp} \subset F$ (c) Lagrangian if $F = F^{\perp}$
(d) Symplectic if $F \cap F^{\perp} = \{0\}$.

Theorem 4.1. Every finite-dimensional symplectic vector space has even dimension and contains a lagrangian subspace.
Proof: Let F be a k-dimensional isotropic subspace of (E, ω). Then

$$F \subset F^{\perp}$$
$$\Rightarrow\; dim\, F \le dim F^{\perp}$$
$$\Rightarrow\; dim\, F \le dim\, V - dim\, F$$
$$\Rightarrow\; 2\, dim\, F \le dim\, V \;\Rightarrow\; dim\, F \le \frac{1}{2} dim\, V$$

If $dim\, F \neq \frac{1}{2} dim\, V$ then $F \neq F^{\perp}$ and so we can construct an isotropic subspace F' of dimension k+1. By continuing in this way, we construct a sequence of isotropic subspace of increasing dimension. The sequence can certainly be started and it must terminate at the point for which the subspace has dimension $k = \frac{1}{2}\, dim\, V$ and is Lagrangian. This completes the proof.

□

Definition 4.5. The pair (M, ω) of a smooth manifold M with a symplectic form ω is called a symplectic manifold.

Theorem 4.2. (Darboux-Weinstein theorem) [4]
Let (M, ω) be a symplectic manifold. Then for any $p \in M$, there exists a local chart $(U, x_1, x_2, \dots, x_n, , y_1, y_2, \dots, y_n)$ around p so that on U,

$$\omega|U = \sum_{i=1}^{n} dx_i \wedge dy_i$$

Theorem 4.3. (De Rham's Theorem) [5]
De Rham's theorem, proved by *Georges de Rham* in 1931, states that, “The De Rham cohomological group $H_{DR}^{k}(M)$ is isomorphic to the singular cohomo-logical group $H^k(M, \mathbb{R})$ with real coefficients”.

5. Comparison between symplectic and Riemannian geometry

Symplectic Geometry	**Riemannian Geometry**
1. <u>Symplectic Manifolds</u> $(\boldsymbol{i})$ Symplectic manifold is a pair (M, ω), where: (a) $\omega \in \Omega^2(M)$ i.e., $\omega(Y, X) = -\omega(X, Y)$ $\omega(fX + gY, Z) = f\omega(X, Z) + g\omega(Y, Z)$ (b) ω is nondegenerate, i.e.: $\omega(X, Y)) = 0, \forall X \in \chi(M) \Leftrightarrow Y = 0$ (c) ω is closed, i.e. $d\omega = 0$. We call ω a symplectic form. $(\boldsymbol{ii})$ If any manifold holds the following necessary conditions, that manifolds qualify for being symplectic. Necessary conditions: (N1) $dim\, M = 2n$ (N2) M is oriented (N3) if M is compact then $H_{DR}^{2}(M, \mathbb{R}) \neq 0$	**1. <u>Riemannian Manifolds</u>** $(\boldsymbol{i})$ Riemannian manifold is a pair $(M, <,>)$, where: (a) $<,>: \chi(M) \times \chi(M) \to C^{\infty}(M)$ satisfies: $< Y, X > = < X, Y >$ $< fX + gY, Z > = f < X, Z > + g < Y, Z >$ (b) $<,>$ is positive definite. Consequence: $<,>$ is non-degenerate. $(\boldsymbol{ii})$ All smooth manifolds qualify for being Riemannian.
2. <u>Examples</u> $(\boldsymbol{i})$ $(\mathbb{R}^{2n}, \omega_0)$ gives an example of non-compact symplectic manifold. Here, $\omega_0 = \sum_{i=1}^{n} dx_i \wedge dy_i$ is the standard symplectic structure and$(x_1, x_2, \dots, x_n, , y_1, y_2, \dots, y_n)$ are the coordinates of $\mathbb{R}^{2n}$.	**2. <u>Examples</u>** $(\boldsymbol{i})$ Euclidean spaces are also Riemann spaces or Riemannian manifold. So, $(\mathbb{R}^n, < X, Y >)$ gives an example of Riemannian manifold. Here, $\mathbb{R}^n$ with canonical co-ordinates $\{x^i\}$ and with metric $g = (dx^1)^2 + (dx^2)^2 + \cdots + (dx^n)^2$ $g = \|\| g^{ik} \|\| = diag\,[1, 1, \dots, 1]$

	This is a basis example of *n*-dimensional Euclidean space, where scalar product is defined by the formula: $g(X,Y) = \langle X,Y\rangle = g^{ik} X^i Y^k = X^1Y^1 + X^2Y^2 + \cdots + X^nY^n$.
$(\boldsymbol{ii})$ The $2n$ dimensional torus $T^{2n} = \mathbb{R}^{2n} / \mathbb{Z}^{2n}$ with its standard form $$\omega_0 = \sum_{j=1}^{n} dx_j \wedge dy_j$$ is a symplectic manifold.	$(\boldsymbol{ii})$ $(\mathbb{R}^2, < X, Y >)$ gives another example of Riemannian manifold. Here $\mathbb{R}^2$ with polar coordinates in the domain $y > 0$ $(x = r\cos\varphi, y = r\sin\varphi)$: $dx = \cos\varphi dr - r\sin\varphi d\varphi$, $dy = \sin\varphi dr + r\cos\varphi d\varphi$. In new coordinates the Riemannian metric $g = dx^2 + dy^2$ will have the following appearance: $g = (dx)^2 + (dy)^2 = dr^2 + r^2 (d\varphi)^2$ We see that for matrix $g = \|\ g^{ik}\ \|$ Scalar product is defined by the formula: $g(X,Y) = \langle X,Y\rangle = g^{ik} X^i Y^k = X^1Y^1 + r^2X^2Y^2$
3. <u>Special Vector Fields</u> $(\boldsymbol{i})$ Non-degeneracy of ω implies that the following is an isomorphism: $I: \chi(M) \to \Omega^1(M)$, $X \to i_X\omega = \omega(X,\cdot)$ Given $f \in C^\infty(M)$, its Hamiltonian vector field is: $X_f = I^{-1}(df)$ $(\boldsymbol{ii})$ Hamiltonian vector field X_f is tangent to the level surface: $\Sigma_C = \{p \in M: f(p) = C\}$ Other important vector field: A vector field X is said to be symplectic if $I(X)$ is closed. In other words: $di_X\omega = 0$ Hence, Hamiltonian vector fields are symplectic, since $d(df) = 0$.	**3. <u>Special Vector Fields</u>** $(\boldsymbol{i})$ Non-degeneracy of <,> implies that the following is an isomorphism: $I: \chi(M) \to \Omega^1(M)$, $X \to < X,\cdot >$ Given $f \in C^\infty(M)$, its gradient vector field is: $\nabla f = I^{-1}(df)$ $(\boldsymbol{ii})$ gradient vector field ∇f is normal to the level surface: $\Sigma_C = \{p \in M: f(p) = C\}$

4. <u>Equivalence</u> Two symplectic manifolds (M,ω) and (M',ω') are symplectomorphic if there exists a C^1 map: $\varphi : M \to M'$ satisfying: $\varphi^*\omega' = \omega$ i.e., $\omega'\varphi_{(p)}\,(d\varphi_p(X), d\varphi_p(Y)) = \omega_p(X,Y)$ Where, φ is called a symplectic map and necessarily $d\varphi_p$ is injective, for all p so: $dimM \leq dimM'$	**4. <u>Equivalence</u>** Two Riemannian manifolds $(M,<,>)$ and $(M',<,>')$ are isometric if there exists a C^1map: $\varphi : M \to M'$ satisfying: $< d\varphi_p(X), d\varphi_p(Y) >'_{\varphi(p)} = < X,Y >_p$ is called an isometry and necessarily $d\varphi_p$ is injective, for all p so: $dimM \leq dimM'$
5. <u>Invariants</u> Darboux's theorem implies that, there are no local invariants (apart from dimension) in Symplectic Geometry.	**5. <u>Invariants</u>** Curvature is a local invariant in Riemannian Geometry.

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