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The 2-Tuple Domination Problem on Trapezoid Graphs

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Abstract. Given a simple graph G = (V, E) and a fixed positive integer *k*. In a graph G, a vertex is said to dominate itself and all of its neighbors. A set D ⊆ V is called a *k*-tuple dominating set if every vertex in V is dominated by at least *k* vertices of D. The *k*-tuple domination problem is to find a minimum cardinality *k*-tuple dominating set. This problem is NP-complete for general graphs. In this paper, the same problem restricted to a class of graphs called trapezoid graphs is considered. In particular, we presented an $O(n^2)$-time algorithm to solve the 2-tuple domination problem on trapezoid graphs.

Keywords: Design of algorithms, analysis of algorithms, trapezoid graphs, domination, 2-tuple domination.

AMS Mathematics Subject Classification(2010): 05C85, 68R10

1. Introduction

Trapezoid graphs are intersection graphs of the set of trapezoids lying between two horizontal lines and are a class of co-comparability graphs containing both interval graphs and Permutation graphs as a subclass. A trapezoid graph G=(V, E)is a set of trapezoids corresponding to the vertices $i \in$ V and there exists edge(i, j)$\in$ E if and only if the trapezoids i and j intersect with each other. Each trapezoid i has four corner points top left a(i), bottom left b(i), top right c(i), bottom right d(i). It is assumed that no two trapezoids share common end point.

Let T ={1, 2, 3,…, n}, denote the set of trapezoids in the trapezoid diagram for a trapezoid graph G =(V, E)with V=n. For trapezoid i, a(i)< c(i)and b(i)< d(i)holds. The points on each horizontal line of the trapezoid diagram are labeled with distinct integers from 1 to 2n in increasing order from left to right. The terms vertex and trapezoid are interchangeable. In this paper, it is assume that a trapezoid diagram is given and the trapezoids are labeled in increasing order of their top right corner point, i.e. $i < j$ if and only if c(i)<c(j).

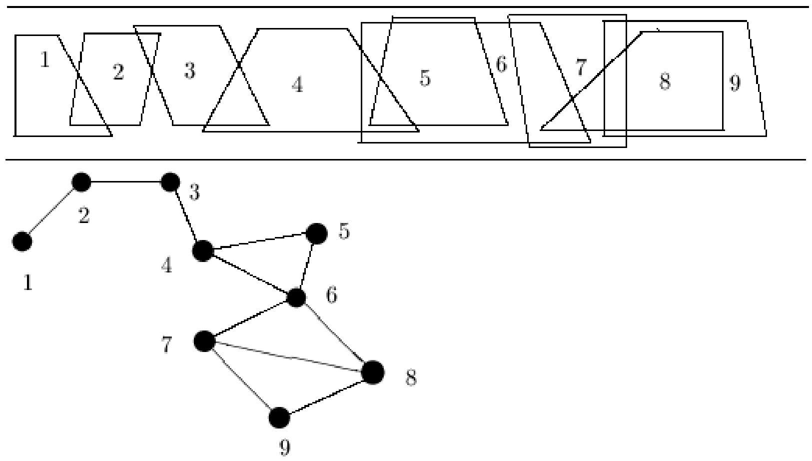

Figure 1 : A trapezoid graph and its trapezoid representation

1.1. Review of previous work

Trapezoid graphs were introduced by Dagan et al. [3]. The fastest algorithm for trapezoid order recognition was proposed by Ma and Spinrad [8] with a running time of O(n^2). The recognition problem for trapezoid graphs was shown by Mertzios and Corneil [9] to succeed in O($n(m+n)$) time. Haynes et al. give detail ideas on the domination problem in graph theory in their two books [6] and [7]. A vertex is said to dominate itself and all its neighbors. A dominating set is a subset D of V such that every vertex in V is dominated by some vertex in D. A set D $\subseteq$ V is called a *k*-tuple dominating set if every vertex in V is dominated by at least k vertices of D where *k* is a fixed positive integer. The *k*-tuple domination number γ_k(G)is the minimum cardinality of a *k*-tuple dominating set of G. If k=2, then the domination problem is called the 2-tuple domination. In 2-tuple domination problem, every vertex in V is dominated by at-least 2 vertex of the dominating set D. Double domination was introduced by Harary and Haynes [5]. In [10], Pramanik, Mondal and Pal solved 2-tuple domination problem on interval graphs using O(n^2) time. Barman, Mondal and Pal solved 2-tuple domination problem on permutation graphs [1]. Other works on trapezoidal graphs are available on [11-13].

1.2. Main result

To the best of our knowledge, no algorithm is available to solve 2-tuple domination problem on trapezoid graph. In this paper, we consider 2-tuple domination problem on trapezoid graph and an O(n^2)time algorithm is designed to solve the problem.

1.3. Organization of the paper

The rest of this paper is organized as follows. Section 2 establishes basic notations and some properties of trapezoid graphs. In Section 3, some lemma and theorem are established. In Section 4, an O(n^2)time algorithm is designed for solving 2-tuple domination problem on a trapezoid graphs and a proof of correctness of the algorithm is provided. The time complexity is also calculated in this section. Finally, Section 5 contains some conclusions.

2. Notations and preliminaries

This section presents the preliminaries on which the desired algorithm depends. A trapezoid *i* is left to the trapezoid *j*, if and only if $i < j$. Similarly, a trapezoid *i* is right to the trapezoid *j* if and only if $i > j$.

Let L(*i*)is farthest left trapezoid intersecting trapezoid *i* and less than *i*. L(*i*)=*i*, if such trapezoid does not exist. Similarly, R(*i*)is farthest right trapezoid intersecting trapezoid *i* and greater then *i*. R(*i*)=*i*, if such trapezoid does not exist. The collection of all farthest left trapezoids forms the set FL and the collection of all farthest right trapezoids is the set FR. That is, FL ={L(*i*): $i \in$ V} and FR ={R(*i*) : $i \in$ V}. Let T ={*i*: $i \in$ FL$\cap$ FR}.

For example, in Figure 1, L(1)=1, L(2)=1, L(3)=2, L(4)=3, L(5)=4, L(6)=4, L(7)=6, L(8)=6, L(9)=6 and R(1)=2, R(2)=3, R(3)=4, R(4)=6, R(5)=6, R(6)=8, R(7)=9, R(8)=9, R(9)=9. FL ={1, 2, 3, 4, 6}, FR ={2, 3, 4, 6, 8, 9} and T =FL$\cap$FR ={2, 3, 4, 6}.

In a graph G =(V, E), N(*i*)is the collection of all adjacent vertices of the vertex *i*, *i.e*., N(*i*) ={$j \in$ V:(*i, j*)$\in$E}. The closed neighborhood of *i* is N[*i*] ={*i*}$\cap$ N(*i*).

The left diagonal of the trapezoid *i* is the line segment joining top left point a(*i*) and the bottom right point d(*i*) and is denoted by Ld(*i*). Similarly, the right diagonal of the trapezoid *i* is the line segment joining top right point c(*i*)and bottom left point b(*i*)and is denoted by Rd(*i*).

Let α(*i*)is the total number of members of the dominating set D which are intersected by Ld(*i*)and β(*i*)is the total number of members of the dominating set D which are intersected by Rd(*i*). Let M(*i*)=max(α(i), β(i)), *i*.e. maximum number of members of the dominating set D which are intersected by Ld(*i*)and Rd(*i*).

3. Some results

The following lemma plays an important role.

Lemma 1. If $i< k < j$ and R(*i*)=*k*, L(*j*)=*k* then *k* is a member of the 2-tuple dominating set.

Proof: Since L(*j*)=*k*, *i.e.*, *k* is farthest left trapezoid of *j* intersecting the trapezoid *j*. Hence *k* is adjacent to the trapezoid *j*. Again, since R(*i*)=*k*, *i.e.*, *k* is farthest right trapezoid of *i intersecting* trapezoid *i*, therefore *k* is adjacent to *i*. So *k* is adjacent to both *i* and *j* where $i < k < j$. Hence the trapezoid *k* is a member of the 2-tuple dominating set.

Now, if there exists a finite number of trapezoids between *i* and *j*, then the trapezoid *k* is the longest trapezoid intersecting trapezoids *i* and *j*, therefore the trapezoid *k* must intersect all the trapezoids between *i* and *j*. Obviously, *k* is adjacent to *i* and *j* and all trapezoids between *i* and *j*. Hence *k* is a member of the 2-tuple dominating set.□

Lemma 2. Every member of the set T ={*i*: $i \in$FL$\cap$ FR}, are the member of the 2-tuple dominating set.

Proof: The vertex $k \in$ T, implies L(*i*)=*k*=R(*j*)for some *i*, $j \in$ V. That is both the trapezoid *i* and trapezoid *j* intersect the trapezoid *k*. So, the trapezoid *k* dominates both the trapezoid *i* and *j*. It is obvious that the trapezoid *k* dominates at least three trapezoids including *k*, because there may be more than one left as well as right adjacent to the trapezoid *k*. It is easy to verify that the vertex *k* dominates maximum number of vertices including the vertex *i* and vertex *j*. Since the aim is to find a minimum cardinality dominating set, *k* must be a member of the 2-tuple dominating set. □

Lemma 3. If |N [i] | =2 then all the members of N [i] belongs to the 2-tuple dominating set.

Proof: If |N [i]|=2 then i has exactly one adjacent vertex *i.e.*, N [i] =N(i) ∪{i}. Obviously, i represent a pendent vertex.

Therefore to cover the vertex i by the two member of the dominating set, the vertex i and its one adjacent vertex must be the member of the 2-tuple dominating set. Hence if |N [i] |=2 then the vertex i and its one adjacent vertex are the members of the 2-tuple dominating set D. □

Lemma 4. If |N[i]| =3 then two adjacent trapezoids of i are the members of the 2-tuple dominating set.

Proof: Since |N[i]| =3 then the three cases may arise.

Case 1 : N [i]=LN(i)∪{i} ∪RN(i), here trapezoid i have right as well as left adjacent trapezoids intersecting trapezoid i this implies L(i)is the only member of LN(i)and R(i)is the only member RN(i). Hence right adjacent R(i)and left adjacent L(i)of i dominates i.

Case 2: N [i] =LN(i)∪{i}. Here trapezoid i have two left adjacent trapezoids. This two left adjacent vertices dominates the vertex i.

Case 3: N [i] ={i}∪ RN(i). Here also trapezoid i have two right adjacent trapezoids. So this two right adjacent vertices dominates the vertex i.

Hence in all the above three cases, i is dominated by its two adjacent trapezoids. Thus two adjacent trapezoids are the members of the 2-tuple dominating set D. □

Lemma 5. If $i \in$ V \ D and M(i)=max(α(i), β(i))=1 then the trapezoid i is the member of the 2-tuple dominating set.

Proof: Since M(i)=max(α(i), β(i)) =1, where $i \in$ V \D, then any one of the member of the dominating set D intersect by Ld(i)or Rd(i). That means i is covered by only one member of the dominating set. Hence by the definition of 2-tuple domination, i must be a member of the 2-tuple dominating set D. □

Lemma 6. If M(i)=max(α(i), β(i))> 2 and M(N(i))> 2, where $i \in$ V, then the trapezoid i does not belong to the 2-tuple dominating set.

Proof: If M(i)==max(α(i), β(i))>2 and M(N(i))> 2, where $i \in$V then at least three members of the dominating set D are covered the vertex i. Also all the adjacent vertices of the vertex i are also covered by at least three vertices of the dominating set D. So if the vertex i belongs to the dominating set then to get the minimum 2-tuple dominating set, we eliminate the vertex i from the dominating set. Hence i is not a member of the 2-tuple dominating set if M(i)=max(α(i), β(i))> 2 and M(N(i)) > 2, where $i \in$ V. □

4. Description of algorithm

The strategy of the proposed algorithm is as follows.

For all $i \in$ V, we have to compute farthest left as well as farthest right trapezoids intersecting each trapezoid i. All the farthest left trapezoids intersecting each trapezoid i forms the set FL and similarly all the farthest right trapezoids intersecting each trapezoid i forms the set FR.Next, we construct the sets T =FL ∩ FR. Initially, T is taken as 2 tuple dominating set D. To select the next members of the dominating set D, we have to compute the value of |N[i]| for all $i \in$ V \ D. Next we compute M(i), for all $i \in$ V \ D. There are two cases which may arise.

Case 1: If M(i)=0 then vertex i and one of its adjacent vertex are the members of the dominating set.

Case 2: If M(i)=1 then the vertex i is a member of the dominating set D. At the end of our algorithm we again compute M(i) for all $i \in$ V. If both M(i)> 2 and M(N(i))> 2 then to get minimum 2-tuple dominating set D, we eliminate i from D, otherwise D remains unaltered.

4.1. The algorithm

A formal description of the algorithm is given below.

Algorithm 2TDP

Input: A trapezoid graph G =(V, E).

Output: A minimum cardinality 2-tuple dominating set of G.

Step 1: Compute L(i)and R(i)for each vertex $i \in$ V. Compute FL and FR.

Step 2: Construct the set T =FL∩FR. Initialize D =T.

Step 3: Compute N [i], N(i), LN(i), RN(i)for each vertex $i \in$ V \D

Step 4: If |N[i] | =2 then D =T∪ N[i] [By Lemma 3]

Step 5: If |N[i] | =3 then D =D∪ N(i) [By Lemma 4]

Step 6: Compute α(i), β(i), and M(i)for each vertex $i \in$ V \ D.

Step 7: If M(i)=0 for each $i \in$ V \ D.

then D =D∪ {i}∪ L(i) or D =D ∪{i}∪R(i).

else if M(i)=1 for each vertex $i \in$V \ D then D =D∪{i} [By Lemma 5]

else if M(i)=2 for each vertex $i \in$V \ D

Compute α(i), β(i), and M(i) for each vertex $i \in$ V

if M(i)> 2 and M(N(i)))> 2 then D =D \{i} [By Lamma 6]

endif.

Lemma 9. The set D is a minimum cardinality 2-tuple dominating set.

Proof: Let m_1, m_2, m_3, ..., m_k are the members of the dominating set D obtained by the algorithm 2-TDP. We have to prove that D is minimum 2-tuple dominating set.

If possible, let, there exist D′ ⊂ D such that D′ is a 2-tuple dominating set. Since D′ ⊂ D, there must exists at least one member of D, say m_i, such that $m_i \not\subset$ D′.

Case 1: If M(m_i)=2 where $m_i \in$D, then m_i is covered by itself and one of the adjacent vertices of m_i. Now since $m_i \notin$ D′, therefore M(m_i)=1 with respect to the dominating set D′. Here the vertex m_i is covered by only its adjacent vertex, a contradiction.

Case 2: If M(m_i)> 2 where $m_i \in$ D, then there may exist at least one vertex m_j , say, such that M(m_j)=2. Now since $m_i \notin$D′, the value of M(m_j) is 1 with respect to the dominating set D′. This is also a contradiction. Hence the result follows.□

Theorem 1. Algorithm 2-TDP finds a 2-tuple dominating set on trapezoid graphs in O(n^2)time.

Proof: The time complexity of algorithm 2-TDP is caused mainly by the computation of L(i), R(i), N(i)and M(i). For each $i \in$ V, calculation of L(i)and R(i)requires O(n^2)time where n is the total number of trapezoids. Calculation of FL and FR takes O(n)time each. For each $i \in$ V, calculation of N(i)takes O(n)times. This is repeated for n times. Therefore the total time to compute N(i)is O(n^2). M(i)can be calculated in O(n^2)time. In the last step, calculation of M(N(i))requires O(n)time. Thus the overall time complexity is O(n^2)+ O(n^2)+ O(n)+ O(n)+ O(n^2)+ O(n^2)+ O(n)=O(n^2).□

5. Concluding remarks

In this paper, we developed an algorithm that solves the minimum cardinality 2-tuple domination problem on trapezoid graphs using O(n^2)time. The same algorithm can be applied to a subset of trapezoid graphs known as interval graphs and permutation graphs and the time complexity remains unchanged. A future study can investigate to design a polynomial time algorithm to solve k-tuple domination problem on trapezoid graphs for k> 2.

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