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Epimorphisms in the Category of Hausdorff Fuzzy Topological Spaces

Seema Mishra

Department of Mathematical Sciences, I.I.T. (B.H.U.), Varanasi-221005, India
Email: seemamishra.rs.apm12@itbhu.ac.in

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Abstract. In this paper we have characterized the epimorphisms in the full subcategory FTS_2 of Hausdorff fuzzy topological spaces (introduced by Srivastava et al.[10]) of the category *FTS* of fuzzy topological spaces and fuzzy continuous functions using the Salbany- type closure operator.

Keywords: Fuzzy topological space, Salbany-type closure, Hausdorff fuzzy topological space

AMS Mathematics Subject Classification (2010): 54A40, 16B50

1. Introduction

Salbany introduced a special type of closure operator in [9]. Alderton and Castellini [2] had used Salbany-type closure operator to characterize the epimorphisms in three categories of separated fuzzy topological spaces viz., the categories FTS_0 of $0^* - T_0$ fuzzy topological spaces (introduced by Wuyts and Lowen [12]) and fuzzy continuous functions, FTS_1 of FT_s−fuzzy topological spaces (introduced by Ghanim, Kerre and Mashhour [5]) and fuzzy continuous functions and $FTS_{\alpha 2}$ of αT_2−fuzzy topological spaces (introduced by Rodabaugh [8]) and fuzzy continuous functions. All these three categories FTS_0 , FTS_1 , $FTS_{\alpha 2}$ are full subcategories of the category *FTS* of fuzzy topological spaces and fuzzy continuous functions. In this paper, we have used Salbany-type closure operator to characterize the epimorphisms in the full subcategory FTS_2 of Hausdorff fuzzy topological spaces (introduced by Srivastava et al.[10]), of *FTS*.

2. Preliminaries

Definition 1. [13] A *fuzzy set A* in a non empty set *X* is a function from *X* to the closed unit interval [0, 1] i.e., $A: X \rightarrow [0, 1]$. Now we define some basic fuzzy set operations as follows:

Let *A* and *B* be fuzzy sets in *X*. Then
(1) $A = B$ if $A(x) = B(x)$, $\forall x \in X$.
(2) $A \subseteq B$ if $A(x) \leq B(x)$, $\forall x \in X$.
(3) $(A \cup B)(x) = \max\{A(x), B(x)\}$, $\forall x \in X$.
(4) $(A \cap B)(x) = \min\{A(x), B(x)\}$, $\forall x \in X$.
(5) $A^c(x) = 1 - A(x)$, $\forall x \in X$ (here A^c denotes the complement of *A*).

Definition 2. [7] Let Ω be an index set and $\{A_i : i \epsilon\ \Omega\}$ be a family of fuzzy sets in X. Then their union $\bigcup_{i\epsilon\,\Omega} A_i$ and intersection $\bigcap_{i\epsilon\,\Omega} A_i$ are defined respectively as follows:
(1) $(\bigcup_{i\epsilon\,\Omega} A_i)(x) = \sup\{A_i(x) : i\epsilon\ \Omega\},\ \forall\, x \in X$.
(2) $(\bigcap_{i\epsilon\,\Omega} A_i)(x) = \inf\{A_i(x) : i\epsilon\ \Omega\},\ \forall\, x \in X$.

Definition 3. [7] A *fuzzy topological space* is a pair (X, τ) consisting of a non empty set X and a family τ of fuzzy sets in X satisfying the following conditions:
(1) $\phi, X \in \tau$;
(2) If $A, B \in \tau$, then $A \cap B \in \tau$;
(3) If $A_i \in \tau, \forall\, i\epsilon\ \Omega$, where Ω is an index set, then $\bigcup_{i\epsilon\,\Omega} A_i \in \tau$.
Here ϕ, X respectively denote the constant fuzzy sets in X taking values 0 and 1, τ is called a fuzzy topology on X and members of τ are called $\tau-$open fuzzy sets. A fuzzy set A in X is called $\tau-$closed if $A^c \in \tau$.

Definition 4. [13] Let $f : X \to Y$ be a function and U, V be fuzzy sets in X and Y respectively. Then $f(U)$ and $f^{-1}(V)$ are fuzzy sets in Y and X respectively, defined as follows:

$$f(U)(y) = \begin{cases} \sup_{x\in f^{-1}(y)} U(x), & \text{if } f^{-1}(y) \neq \phi \\ 0, & \text{otherwise} \end{cases}$$

and $$f^{-1}(V)(x) = V(f(x)),\ \forall\, x \in X.$$

Definition 5. [4] Let (X, τ) and (Y, σ) be fuzzy topological spaces. Then a function $f : (X, \tau) \to (Y, \sigma)$ is said to be *fuzzy continuous* if $f^{-1}(V) \in \tau$ whenever $V \in \sigma$.

The category of fuzzy topological spaces and fuzzy continuous functions will be denoted by *FTS*.

Definition 6. [1] Let $f, g : A \longrightarrow B$ be a pair of morphisms. A morphism $E \overset{e}{\to} A$ is called an *equalizer* provided that the following conditions hold:
(1) $f\ o\ e = g\ o\ e$;
(2) For any morphism $e' : E' \longrightarrow A$ with $f\ o\ e' = g\ o\ e'$, there exists a unique morphism $\bar{e} : E' \longrightarrow E$ such that $e' = e\ o\bar{e}$.

Remark 7. [2] In *FTS*, the equalizer between a pair of fuzzy continuous functions $f, g : (X, \tau) \longrightarrow (Y, \sigma)$ can be identified with the set $K[f, g] = \{x \in X |\ f(x) = g(x)\}$.

Definition 8. [9] Let (X, τ) be a fuzzy topological space and Λ be a class of *FTS* objects. Then the *Salbany-type closure* of a subset M of X with respect to Λ, denoted by $[M]_\Lambda$, is the intersection of all the equalizers between a pair of fuzzy continuous functions from (X, τ) to some Λ-object (Y, σ) that agrees on M i.e,

$$[M]_\Lambda = \cap\{K[f, g] |\ K[f, g] \supset M \text{ where } f, g : (X, \tau) \longrightarrow (Y, \sigma),\ (Y, \sigma) \in \Lambda\}.$$

The above closure operator is idempotent which is already proved in [9].

Definition 9. [11] A *fuzzy point* x_λ $(0 < \lambda < 1)$ in X, is a fuzzy set in X given by

$$x_\lambda(x') = \begin{cases} \lambda, & \text{if } x' = x \\ 0, & \text{otherwise} \end{cases}$$

Here x and λ are respectively called the *support* and *value* of x_λ. The fuzzy point x_λ in X is said to *belong to* a fuzzy set A in X if $\lambda < A(x)$ (cf.[10]).
Two fuzzy points x_r and y_s are said to be *distinct* if $x \neq y$.

Definition 10. [7] A *fuzzy singleton* x_λ ($0 < \lambda \leq 1$) in X, is a fuzzy set in X given by

$$x_\lambda(x') = \begin{cases} \lambda, & \text{if } x' = x \\ 0, & \text{otherwise} \end{cases}$$

Here x and λ are respectively called the *support* and *value* of x_λ. The fuzzy singleton x_λ in X is said to *belong to* a fuzzy set A in X if $\lambda \leq A(x)$.

Definition 11. [7] A fuzzy singleton x_λ in X is said to be *quasi-coincident* with a fuzzy set A in X if $\lambda > A^c(x)$ or $\lambda + A(x) > 1$.

Definition 12. [7] Let A and B be fuzzy sets in X. Then A is said to be quasi-coincident with B, denoted by AqB if there exists $x \in X$ such that $A(x) > B^c(x)$ or $A(x) + B(x) > 1$. In this case A and B are said to be quasi-coincident (with each other) at x.

Definition 13. [7] Let (X, τ) be a fuzzy topological space and A be a fuzzy set in X. Then A is said to be a Q-neighborhood of x_λ if there exists $B \in \tau$ such that $x_\lambda q B \subseteq A$.

Proposition 14. [7] $A \subseteq B$ iff A and B^c are not quasi-coincident.

3. Epimorphisms in the category of Hausdorff fuzzy topological spaces

We first state the following theorem given in [2] which is a special case of Theorem 1.11 of [3]. This theorem will be used in proving the main result of this paper.

Theorem 1. Let Λ be a subcategory of *FTS* and $(X,\tau) \xrightarrow{f} (Y,\sigma)$ be a morphism in Λ. Then f is an epimorphism in Λ iff $[f(X)]_\Lambda = Y$.

Definition 2. [7] Let (X,τ) be a fuzzy topological space and M be a fuzzy set in X. Then the intersection of all the $\tau-$closed fuzzy sets containing M is called the *closure* of M, denoted by $\bar{M}$.

M is $\tau-$closed iff $\bar{M} = M$.

Theorem 3. [7] A fuzzy singleton $x_\lambda \in \bar{M}$ iff each Q-neighborhood of x_λ is quasi-coincident with M.

Definition 4. [7] A fuzzy singleton x_λ in X is called an *adherence point* of a fuzzy set M in X if every Q-neighborhood of x_λ is quasi-coincident with M.

Corollary 5. [7] $\bar{M}$ is the union of all adherence points of M i.e., $\bar{M} = \cup \{x_\lambda : U \in \tau, U(x) > 1 - \lambda \Longrightarrow \text{there exists } a \in X \text{ such that } U(a) + M(a) > 1\}$

From Corollary 5, we get the following proposition:

Proposition 6. If M is a crisp subset of X. Then M is τ−closed iff $\forall\, x \in X - M$, there exists $U \in \tau$ such that $U(x) > 1 - \lambda$ and $U(a) = 0$, $\forall\, a \in M$.
Proof: $M \subseteq X$ is τ −closed iff $M = \bar{M} = \cup\{x_\lambda :\ U \in \tau, U(x) > 1 - \lambda \Longrightarrow$ there exists $a \in X$ such that $U(a) + M(a) > 1\}$. Now if $x \in X - M$, then $M(x) = 0$, therefore $x_\lambda \notin M$ and hence $x_\lambda \notin \bar{M}$ implying that there exists $U \in \tau$ such that $U(x) > 1 - \lambda$ and $U(a) + M(a) \leq 1, \forall\, a \in X$. Now $M(a) = 1, \forall a \in M,$ hence $U(a) = 0, \forall a \in M$.

Definition 7. [10] A fuzzy topological space (X,τ) is said to be Hausdorff or FT_2 if for every distinct pair of fuzzy points $x_r,\ y_s$ in X, there exist $U, V \in \tau$ such that $x_r \in U$, $y_s \in V$ and $U \cap V = \phi$.

From now onwards, the full subcategory of FTS consisting of all FT_2 fuzzy topological spaces will be denoted by FTS_2.

Proposition 8. [10] Let $f, g : (X,\tau) \rightarrow (Y,\sigma)$ be fuzzy continuous functions, where $(Y,\sigma) \in FTS_2$. Then the set $\{x \in X \mid f(x) = g(x)\}$ is τ−closed in X.

Proposition 9. Arbitrary intersections of τ −closed sets are τ −closed.
The proof follows from the condition (3) of definition 3 (Section 2).

Proposition 10. Let (X,τ) be a FT_2space and $M \subseteq X$. Then $\bar{M} \subseteq [M]_{FTS_2}$.
Proof. From Propositions 8 and 9, we obtain that $[M]_{FTS_2}$ is τ−closed and $M \subseteq [M]_{FTS_2}$. So we have $\bar{M} \subseteq [M]_{FTS_2}$.
For the following definition and result, we refer to ([2], [6])

Definition 11. [2] Let $\{(X_i, \tau_i)\}_\Omega$, where Ω is an index set, be a family of fuzzy topological spaces and for each $i \in \Omega$, we have a function $X_i \xrightarrow{f_i} X$. Then the fuzzy topology over X which is final with respect to the family $\{(X_i, \tau_i) \xrightarrow{f_i} X\}_\Omega$ is given by $\tau = \{A\colon X \rightarrow I \mid f_i^{-1}(A) \in \tau_i, \forall\, i \in \Omega\}$.

In particular, let (Y,σ) be a fuzzy topological space. Then the fuzzy topology $\sigma \sqcup \sigma$ on $Y \sqcup Y$, the usual disjoint union, which is final with respect to the family of injections $\{\ (Y,\sigma) \xrightarrow{\mu_1} Y \sqcup Y,\ (Y,\sigma) \xrightarrow{\mu_2} Y \sqcup Y\ \}$ is given by

$$\begin{aligned}\sigma \sqcup \sigma &= \{A\colon Y \sqcup Y \longrightarrow I \mid \mu_1^{-1}(A) \in \sigma \text{ and } \mu_2^{-1}(A) \in \sigma\}\\ &= \{U \sqcup V\colon U,\ V \in \sigma\},\end{aligned}$$

where $U \sqcup V$ is defined by

$$(U \sqcup V)(y,i) = \begin{cases} U(y), & \text{if } i = 1 \\ V(y), & \text{if } i = 2 \end{cases}$$

Next, we define a quotient fuzzy topological space which will be needed in the next proposition.

Let (Y,σ) be a fuzzy topological space and $M \subseteq Y$. Define a relation on $Y \sqcup Y$ by

$$(x, i) \sim (y, j) \Longleftrightarrow (x, i) = (y, j) \text{ or } x = y \in M.$$

Clearly, the above relation is an equivalence relation. Let Q be the quotient set of $Y \sqcup Y$ with respect to the above equivalence relation and having the fuzzy topology ζ which is final with respect to the quotient map

$$(Y \sqcup Y, \sigma \sqcup \sigma) \xrightarrow{q} Q$$

where q is given by

$$q(x,i) = \begin{cases} \{(x,1),(x,2)\}, & \text{if } x \in M \\ \{(x,i)\}, & \text{if } x \notin M \end{cases}$$

Hence,

$$\zeta = \{A: Q \longrightarrow I \mid q^{-1}(A) \in \sigma \sqcup \sigma\}$$
$$= \{ A: Q \longrightarrow I \mid Aoq = V_1 \sqcup V_2 \text{ for some } V_1, V_2 \in \sigma\}.$$

Lemma 12. [2] Let U_1 and U_2be arbitrary functions from Y to I. Then $q^{-1}\big(q(U_1 \sqcup U_2)\big) = U_1 \sqcup U_2$ iff $U_1(a) = U_2(a)$, $\forall a \in M$.
We now give the following proposition, the proof of which is on similar lines as in [2].

Proposition 13. For each FT_2 space (Y,σ) and each $\tau-$closed set M of Y, there exists a FT_2 space (Z,κ) and fuzzy continuous functions $f, g : (Y,\sigma) \longrightarrow (Z,\kappa)$ such that $M = \{y \in Y \mid f(y) = g(y)\}$.
Proof: Let (Z,κ) be the quotient space (Q,ζ) defined above. First we show that $(Q,\zeta) \in FTS_2$. To show this, suppose that $q(x,i)_r$ and $q(y,j)_s$ are two distinct fuzzy points in (Q, ζ). Then consider the following cases:
Case I: $x \neq y$.
Since (Y,σ) is a FT_2space, $\exists U, V \in \sigma$ such that

$$x_r \in U,\ y_s \in V \text{ and } U \cap V = \phi$$
$$\Rightarrow r < U(x),\ s < V(y) \text{ and } U \cap V = \phi. \qquad (1)$$

Set $U_1' = q(U \sqcup U)$ and $U_2' = q(V \sqcup V)$.
Then U_1' and U_2' both belong to ζ, by Lemma 12. Further, using equation (1), we have

$$U_1'\big(q(x,i)\big) = q(U \sqcup U)\big(q(x,i)\big) = U(x) > r,$$
$$U_2'\big(q(y,j)\big) = q(V \sqcup V)\big(q(y,j)\big) = V(y) > s.$$

Also, by using equation (1),

$$(U_1' \cap U_2')\big(q(z,i)\big) = \min\{U_1'\big(q(z,i)\big), U_1'\big(q(z,i)\big)\}$$
$$= \min\{U(z), V(z)\} = 0, \ \ \forall\, q(z,i) \in Q$$

showing that $U_1' \cap U_2' = \phi$.
Case II: $x=y$ (say, z)

Since $q(x,i) = \begin{cases} \{(x,1),(x,2)\}, & \text{if } x \in M \\ \{(x,i)\}, & \text{if } x \notin M \end{cases}$, $q(y,j) = \begin{cases} \{(y,1),(y,2)\}, & \text{if } y \in M \\ \{(y,j)\}, & \text{if } y \notin M \end{cases}$

and $q(x,i) \neq q(y,j)$, so $i \neq j$ and $z \notin M$.
Next, since M is $\tau -$ closed, so $\forall\, z \in Y - M$, there exists $U \in \sigma$ such that $U(z) > 1 - \lambda(= t$, say) and $U(a) = 0$, $\forall\, a \in M$. Set $U_1' = q(U \sqcup \phi)$ and $U_2' = q(\phi \sqcup U)$. By Lemma 12, U_1' and U_2' both belong to ζ and

$$U_1'\big(q(z,1)\big) = q(U \sqcup \phi)\big(q(z,1)\big) = U(z) > 1 - \lambda(= t),$$
$$U_1'\big(q(z,2)\big) = q(\phi \sqcup U)\big(q(z,2)\big) = U(z) > 1 - \lambda(= t).$$

Also, we have if $d \notin M$,

$$(U_1' \cap U_2')\big(q(d,1)\big) = \min\{U_1'\big(q(d,1)\big), U_2'\big(q(d,1)\big)\} = \min\{U(d), 0\} = 0$$

and $(U_1' \cap U_2')\big(q(d,2)\big) = \min\{U_1'\big(q(d,2)\big), U_2'\big(q(d,2)\big)\} = \min\{0, U(d)\} = 0.$
Next, if $d \in M$, then

$$(U_1' \cap U_2')\big(q(d,i)\big) = \min\{U_1'\big(q(d,i)\big), U_2'\big(q(d,i)\big)\}$$

$= \min \{\max \{ U(d), 0\}, \max \{0, U(d)\}\}$
$= U(d) = 0.$

Finally, if we set $f = q\ o\ \mu_1$ and $g = q\ o\ \mu_2$. Then it is easy to verify that $M = \{y \in Y \mid f(y) = g(y)\}$.

Corollary 14. Let (X, τ) be a FT_2 space and $M \subseteq X$. Then $[M]_{FTS_2} = \bar{M}$.
Proof: Since $\bar{\bar{M}} = \bar{M}$, so by previous theorem $\bar{M}$ is an equalizer between a pair of fuzzy continuous functions in FTS_2 and hence we have $[M]_{FTS_2} \subseteq \bar{M}$. Next, using this containment and Proposition 10, we obtain that $[M]_{FTS_2} = \bar{M}$.
Now, by using Theorem 1 for the subcategory FTS_2 of FTS, we obtain a characterization for the epimorphisms in the category FTS_2as follows:

Theorem 15. A fuzzy continuous map $(X, \tau) \xrightarrow{f} (Y, \sigma)$ in FTS_2 is an epimorphism iff $\overline{f(X)} = Y$.

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