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Some Results on Fuzzy Numbers

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Abstract. In this paper, we will make a brief survey on fuzzy numbers and their properties. The equivalence relation between two fuzzy numbers have been investigated with the help of definition given by Puri and Raelscu.

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1. Introduction

In most of cases in our life, the data obtained for decision making are only approximately known. In 1965, Zadeh [10] introduced the concept of fuzzy set theory to meet those problems. In 1978, Dubois and Prade [1,2] defined any of the fuzzy numbers as a fuzzy subset of the real line. Fuzzy numbers allow us to make, the mathematical model of linguistic variables or fuzzy environment. A fuzzy number is a quantity, whose value is imprecise, rather than exact as in the case with "ordinary"(single-valued) numbers. Any fuzzy number can be thought of, as a function whose domain is a specified set. In many respects, fuzzy numbers depict the physical world more realistically than single-valued numbers. The fuzzy numbers and fuzzy values are widely used in engineering applications (especially communications) and experimental sciences because of their suitability for representing uncertain information. In this paper, we will make a brief survey on fuzzy numbers and their arithmetic and algebraic properties. Further, we consider the set of fuzzy numbers, as defined by Puri and Ralescu [5,6] define an equivalence relation therein and consider the equivalence classes as "the fuzzy numbers".

2. Fuzzy number

Definition 2.1. A fuzzy set A in R (real line) is defined to be a set of ordered pairs, $A=\{(x,\mu_A(x))/x\in R\}$ where $\mu_A(x)$ is called the membership function for the fuzzy set.

Definition 2.2. The **α-cut** of α-level set of fuzzy set A is a set consisting of those elements of the universe X whose membership values exceed the threshold level α.
That is $A_\alpha=\{x\in X/\mu_A(x)\geq\alpha\}$

Definition 2.3. A fuzzy set A is called **normal**, if there is at least one point $x \in R$ with $\mu_A(x) = 1$

Definition 2.4. A fuzzy set A on R is **convex,** if for any $x, y \in R$ and any $\lambda \in [0,1]$ we have $\mu_A(\lambda_x + (1-\lambda)y) \geq \min\{\mu_A(x), \mu_A(y)\}$

Definition 2.5. A **fuzzy number** is a fuzzy set on the real line that satisfies the conditions of normality and convexity.

A fuzzy number which is normal and convex is referred to as a **normal convex fuzzy number.**

Definition 2.6. If a fuzzy set is convex and normalized and its membership function is defined in R and piecewise continuous, it is called as "**fuzzy number**".
Fuzzy number represents a real number whose boundary is fuzzy.

Definition 2.7. [4] A fuzzy number A is called **positive**, denoted by $A > 0$, if its membership function $\mu_A(x)$ satisfies $\mu_A(x) = 0$, $\forall\ x \leq 0$

Definition 2.8. [4] A fuzzy number A is called **non-negative**, denoted by $A \geq 0$, if its membership function $\mu_A(x)$ satisfies $\mu_A(x) = 0$, $\forall\ x \leq 0$

3. Properties of fuzzy numbers

Definition 3.1. Let A and B be fuzzy numbers in R and let $*$ denote any of the four basic arithmetic operations. Then we define a fuzzy set on R, $A * B$ by the equation

$$(A * B)(z) = \underset{Z=x*y}{Sup} \min[A(x), B(y)], \quad \text{for all } z \in R$$

More specifically, we define for all $z \in R$

$$(A + B)(z) = \underset{Z=x+y}{Sup} \min[A(x), B(y)],$$

$$(A - B)(z) = \underset{Z=x-y}{Sup} \min[A(x), B(y)],$$

$$(A \cdot B)(z) = \underset{Z=x\bullet y}{Sup} \min[A(x), B(y)],$$

$$(A / B)(z) = \underset{Z=x/y}{Sup} \min[A(x), B(y)],$$

Theorem 3.2. Let $* \in \{+, -, \cdot, /\}$, and let A, B denote continuous fuzzy numbers. Then, the fuzzy set $A * B$ defined by **3.1.** is a continuous fuzzy number.

Theorem 3.3. If A and B are convex fuzzy numbers in the real line R, then $A + B$, $A - B$, $A \cdot B$ are also convex fuzzy numbers.

Proof : For each $0 < \alpha \leq 1$, the α-level sets A_α and B_α of convex fuzzy numbers A and B are convex sets(or intervals) in R. Thus, for any α_1 and α_2 with $0 < \alpha_1 \leq \alpha_2 \leq 1$,

$A_{\alpha_2} \subseteq A_{\alpha_1}$ and $B_{\alpha_2} \subseteq B_{\alpha_1}$ Therefore we have, $A_{\alpha_2} + B_{\alpha_2} \subseteq A_{\alpha_1} + B_{\alpha_1}$ and $A_{\alpha_2} \cdot B_{\alpha_2} \subseteq A_{\alpha_1} \cdot B_{\alpha_1}$ which leads to $(A+B)_{\alpha_2} \subseteq (A+B)_{\alpha_1}$ and $(A \cdot B)_{\alpha_2} \subseteq (A \cdot B)_{\alpha_1}$ Further $(A+B)_{\alpha_i}$ and $(A \cdot B)_{\alpha_i}$ are intervals (or convex sets) for each α_i $(i = 1,2)$. Thus, fuzzy numbers $A+B$ and $A \cdot B$ are shown to be convex fuzzy numbers. Next we shall prove the convexity of $A-B$.

Let $-B$ be defined by $0-B,$ then the membership function of $-B$ will be expressed as, $\mu_{-B}(x) = \mu_B(-x), \quad x \in R$ and $-B$ can be easily shown to be convex, if B is convex.

Thus, $A-B$ is proved to be convex, since $A-B$ is expressed as $A+(-B).$

Remark:

1. It should be noted that, for discrete fuzzy numbers, the convexity of $A+B$, $A-B$ and $A \cdot B$ does not hold in general.
2. If B is a zero convex fuzzy number, then $\frac{1}{B}$ $(=1 \div B)$ is not a convex fuzzy number.

Theorem 3.4. If A is a convex fuzzy number and B is a positive (or negative) convex fuzzy number, then $A \div B$ is a convex fuzzy number.

Proof : It will be sufficient to prove that $\frac{1}{B}$ is convex, if B is positive convex.

Since $A \div B$ can be represented as $A \times \left(\frac{1}{B}\right)$. Let x, y, z be real numbers such that, $0 < x \le y \le z,$ then $0 < \frac{1}{z} \le \frac{1}{y} \le \frac{1}{x}$ holds. Thus, we can have $\mu_B\left(\frac{1}{y}\right) \ge \mu_B\left(\frac{1}{z}\right) \wedge \mu_B\left(\frac{1}{x}\right)$ in virtue of the convexity of $B.$ Therefore we can write $\mu_{1/B}(y) \ge \mu_{1/B}(z) \wedge \mu_{1/B}(x)$ which leads to the convexity of $\frac{1}{B}$.

Theorem 3.5. If A and B are normal fuzzy numbers, then $A+B,\ A-B,\ A \cdot B$ and $A \div B$ are also normal.

Remark: For two fuzzy numbers A and B, if the one is convex and the other is non-convex , then the execution results of A and B under $+,\ -,\ \cdot.$ and $\div$ may be convex or non-convex.

Theorem 3.6. For any fuzzy numbers $A,\ B$ and $C,$ we have

i) $(A+B)+C = A+(B+C)$

$(A \cdot B) \cdot C = A \cdot (B \cdot C)$ (Associative laws)

ii) $A + B = B + A$

$A \cdot B = B \cdot A$ **(**Commutative laws)

iii) $A + 0 = A$

$A \cdot 1 = A$ (Identity laws)

where 0 and 1 are zero and unity, respectively, in the ordinary sense.

Theorem 3.7. For any fuzzy number A, there exist no inverse fuzzy numbers A' and A'' under $+$ and $\cdot$, respectively, such that $A + A' = 0$, $A \cdot A'' = 1$

Theorem 3.8. For the positive convex fuzzy numbers A, B and C, the distributive laws holds i.e. $A \cdot (B + C) = (A \cdot B) + (A \cdot C)$

Proof: Let α-level sets of positive convex fuzzy numbers A, B and C be $A_\alpha = [a_1, a_2]$, $B_\alpha = [b_1, b_2]$ and $C_\alpha = [c_1, c_2]$, respectively, then each level set is an interval in R and $0 < a_1 \le a_2$, $0 < b_1 \le b_2$, and $0 < c_1 \le c_2$, hold. Thus for each $0 < \alpha \le 1$,

$$[A \cdot (B + C)]_\alpha = A_\alpha \cdot (B_\alpha + C_\alpha)$$
$$= [a_1, a_2] \cdot ([b_1, b_2] + [c_1, c_2])$$
$$= [a_1(b_1 + c_1), a_2(b_2 + c_2)] \quad \text{.........(i)}$$

Now consider, $[(A \cdot B) + (A \cdot C)]_\alpha = (A_\alpha \cdot B_\alpha) + (A_\alpha \cdot C_\alpha)$

$$= ([a_1, a_2] \cdot [b_1 \cdot b_2]) + ([a_1, a_2] + [c_1, c_2])$$
$$= [a_1 b_1 + a_1 c_1, a_2 b_2 + a_2 c_2]$$
$$= [a_1(b_1 + c_1), a_2(b_2 + c_2)]$$
$$= [A \cdot (B + C)]_\alpha \quad \text{.........(ii)}$$

Therefore, we have $A \cdot (B + C) = (A \cdot B) + (A \cdot C)$

Note that, when α-level set is an empty set ϕ, the following holds,

$A_\alpha + \phi = \phi$; $A_\alpha \cdot \phi = \phi$.

4. Equivalence relation between two fuzzy numbers

Definition 4.1. [5,6] A fuzzy subset A of R is called a fuzzy number, if it satisfies the following conditions,

i) A is an upper semi continuous map

ii) $A_{[a]}$ is non- empty for all a,

iii) $A_{[0]}$ is a bounded subset of R

iv) A is convex.

Definition 4.2. [5,6] : Let A and B be two fuzzy numbers, then we define $A \sim B$, if $(A - B)(c) = 1$, $c = 0$ and $(A - B)(c) = (A - B)(-c)$, $c \ne 0$.

Theorem 4.3. The above relation $\sim$ is an equivalence relation.

Proof: 1.Reflexivity ($A \sim A$)

To prove that, $(A-A)(c)=1,$ if $c=0$ and $(A-A)(c)=(A-A)(-c),$ if $c\neq 0.$

$$(A-A)(0) = \underset{0=a-b}{Sup}\big(A(a)\wedge A(b)\big) = \underset{a=b}{Sup}\big(A(a)\wedge A(b)\big) = \underset{a}{Sup}\, A(a)=1$$

Now to prove that, $(A-A)(c)=(A-A)(-c),$ if $c\neq 0.$

$$(A-A)(-c)=(A+-A)(-c)=\underset{-c=a+b}{Sup}\big(A(a)\wedge -A(b)\big) = \underset{c=a+b}{Sup}\big(A(-a)\wedge A(b)\big) = \underset{c=b+a}{Sup}\big(A(-b)\wedge A(a)\big) = \underset{c=a+b}{Sup}\big(A(a)\wedge -A(b)\big) = (A-A)(c)$$

2. Symmetry ($A\sim B \Rightarrow B\sim A$)

To prove that, $(A-B)(0)=1$ and $(A-B)(c)=(A-B)(-c), \quad c\neq 0$

$\Rightarrow (B-A)(0)=1$ and $(B-A)(c)=(B-A)(-c), \quad c\neq 0$

$$(A-B)(0)=1=\underset{0=a+b}{Sup}(A(a)\wedge -B(b)) = \underset{0=a-b}{Sup}(A(a)\wedge -B(-b)) = \underset{0=a-b}{Sup}(A(a)\wedge B(b)) = \underset{0=b-a}{Sup}(A(b)\wedge B(a)),\ \text{by symmetry} = \underset{0=a-b}{Sup}(B(a)\wedge -A(-b)) = \underset{0=a+b}{Sup}(B(a)\wedge -A(b)) = (B-A)(0)$$

Given that, $(A-B)(c)=(A-B)(-c),$ …………(i)

$(A-B)(-c)=-(A-B)(c)=(B-A)(c)$ ……..(ii)

$(A-B)(c)=-(A-B)(-c)=(B-A)(-c)$ …….(iii)

From (i), (ii), and (iii), it follows that $(B-A)(c)=(B-A)(-c), \quad c\neq 0$

3. Transitivity To prove that, $A\sim B$ and $B\sim C \Rightarrow A\sim C$

First we will prove $(A-C)(0)=1$

We have $\underset{0=a-c}{Sup}(A(a)\wedge B(c))=1$ and $\underset{0=c-b}{Sup}(B(c)\wedge C(b))=1$

$$\underset{0=a-b}{Sup}(A(a)\wedge C(b)) = \underset{0=(a-c)+(c-b)}{Sup}(A(a)\wedge C(b))$$
$$\geq \underset{0=(a-c)+(c-b)}{Sup}(A(a)\wedge B(c)\wedge B(c)\wedge C(b))$$
$$\geq \underset{a-c=b-c=0}{Sup}(A(a)\wedge B(c)\wedge B(c)\wedge C(b))$$
$$= \underset{a-c=0}{Sup}[A(a)\wedge B(c)]\wedge \underset{b-c=0}{Sup}[B(c)\wedge C(b)]$$
$$= 1 .$$

Next to prove that, $(A-C)(c)=(A-C)(-c), \quad c\neq 0$

$$(A-C)(c) = \underset{c=a-b}{Sup}(A(a)\wedge C(b))$$
$$= \underset{c=(a-t)+(t-b)}{Sup}(A(a)\wedge C(b)) \ ; \ t\in R$$
$$= \underset{c=(a-t)+(t-b)}{Sup}[A(a)\wedge B(t)]\wedge[B(t)\wedge C(b)]$$
$$\leq \underset{c=c_1+c_2}{Sup}[(\underset{c_1=a-t}{Sup} A(a)\wedge B(t))\wedge(\underset{c_2=t-b}{Sup}[B(t)\wedge C(b)]$$
$$= \underset{c=c_1+c_2}{Sup}[A-B)(c_1)\wedge(B-C)(c_2)$$
$$= \underset{c=c_1+c_2}{Sup}[A-B)(-c_1)\wedge(B-C)(-c_2)$$
$$= \underset{c=-c_1-c_2}{Sup}[A-B)(c_1)\wedge(B-C)(c_2)$$
$$= \underset{-c=c_1+c_2}{Sup}[A-B)(c_1)\wedge(B-C)(c_2)$$
$$\leq (A-C)(-c)$$

$(A-C)(c)\leq(A-C)(-c),$ ……….(iv)

Similarly we can prove that $(A-C)(-c)\leq(A-C)(c),$ …….(v)

From (iv) and (v), it follows that, $(A-C)(c)=(A-C)(-c), \quad c\neq 0$

Definition 4.4. The fuzzy number 0 is defined by $0(0)=1, \ 0(c)=0(-c), \ \forall\ c$

Remark; $A\sim B$ if and only if, $A\sim B\sim 0$.

Definition 4.5. Let A and B be two fuzzy numbers.

If the mid-point of $A_{[a]}\leq$ mid-point of $B_{[a]}$, then we say that, $A\leq B$

Definition 4.6. A fuzzy number A is called non-negative, if the mid-point of $A_{[a]}\geq 0$, $\forall a>0$

Proposition 4.7. Addition is compatible with equivalence $\sim$ i.e. if $A_1 \sim B_1$ and $A_2 \sim B_2$, then $A_1 + A_2 \sim B_1 + B_2$.

Proposition 4.8. If A, B and C are non-negative fuzzy numbers, then multiplication is compatible with equivalence $\sim$ i.e. $A \sim B$, then $AC \sim BC$

Proposition 4.9. If A_1, A_2, B_1 and B_2 are non-negative fuzzy numbers, then multiplication is compatible with equivalence $\sim$.
i.e. If $A_1 \sim B_1$ and $A_2 \sim B_2$, then $A_1 A_2 \sim B_1 B_2$.
Proof: Since $A_1 - B_1 \sim 0$ and $A_2 - B_2 \sim 0$.

$$\begin{aligned} A_1 A_2 - B_1 B_2 &= A_1 A_2 - B_1 A_2 + B_1 A_2 - B_1 B_2 \\ &= (A_1 - B_1) A_2 + (A_2 - B_2) B_1 \\ &\sim 0\,A_2 + 0\,B_1 \\ &\sim 0 \end{aligned}$$

i.e. $A_1 A_2 \sim B_1 B_2$

Proposition 4.10. If A is a non-negative fuzzy number, then $A^2 \geq 0$

Notation 4.11. The set of equivalence classes of fuzzy numbers is denoted by $\overline{R}$ and the set of equivalence classes of all non-negative fuzzy numbers is denoted by $\overline{R^+}$. The equivalence class containing fuzzy number A is denoted by $[A]$. The equivalence class containing fuzzy number 0 is denoted by $\overline{0}$, where 0 is defined by,

$$0(0) = 1, \quad 0(c) = 0(-c), \quad \forall c$$

Therefore, we have the following,

$[A] = [B]$ if and only if, $[A] - [B] = \overline{0}$

5. Conclusion

In this paper, we have made a brief survey on Fuzzy numbers with their properties. We have considered the definition of fuzzy number given by Puri and Ralescu to determine the equivalence relation between two fuzzy numbers and finally made an attempt to denote equivalence classes as the fuzzy numbers.

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