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Some Characterizations of Weakly Complemented Meet Semilattices

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Abstract. In this paper the authors study the homomorphisms on directed above meet semilattices. Then they include homomorphism theorem for meet semilattices.They establish some results on homomorphic images of semiprime ideals. They also show that in a 0-distributive semilattice, a map $f: S \to \{\{a\}^{\perp\perp} : a \in S\}$ is a semilattice homomorphism if and only if $f(\{a\}^{\perp}) = \{f(a)\}^{\perp}$. Finally some characterizations of weakly complemented meet semilattices relative to J have been included.

Keywords: Semi prime ideal, 0-distributive semilattice, distributive meet semilattice, meet semilattices homomorphism.

AMS Mathematics Subject Classifications (2010): 06A12, 06A99, 06B10

1. Introduction

Varlet [9] first introduced the concept of 0-distributive lattices. Then many authors including [1,2,3,5,6,7] studied them for lattices and semilattices. A meet semilattice S with 0 is called 0-distributive if for all $a,b,c \in S$ with $a \wedge b = 0 = a \wedge c$ imply $a \wedge d = 0$ for some $d \geq b,c$ [3]. The concept of semi prime ideals of a lattice is introduced in [8]. Recently, Begum and Noor [2] have extended the concept for meet semilattices. An ideal J of a meet semilattice S is called a semi prime ideal if for all $a,b,c \in S$ with $a \wedge b \in J$, $a \wedge c \in J$, imply $a \wedge d \in J$ for some $d \geq b,c$. Hence a meet semilattice S with 0 is called 0-distributive if (0] is a semiprime ideal of S. A meet semi lattice S is called *directed above* if for all $a,b \in S$, there exists $c \in S$ such that $c \geq a,b$. We know that every modular and distributive semilattice have the directed above property. Moreover [3] have shown that every 0-distributive meet semilattice is directed above.

Let S and T be two meet semilattices. A map $f: S \to T$ is said to be a homomorphism if f is a meet-preserving map. That is, for all $a,b \in S$, $f(a \wedge b) = f(a) \wedge f(b)$. A homomorphism is called 0-homomorphism if $f(0) = 0$. A one-to-one homomorphism is called a monomorphism or an embedding. A onto homomorphism is called an epimorphism. If $f: A \to B$ is an epimorphism, we say that *B* is a homomorphic image of *A*. A epimorphism is called an isomorphism if it is a one-to-one map. A homomorphism $f : A \to A$ is called an endomorphism, and an isomorphism $f : A \to A$ is called an automorphism. The meet semilattice *S* and *T* are isomorphic if there exist an isomorphism *f* from *S* to *T*. We denote it symbolically by $S \cong T$.

2. Homomorphism and semi prime ideals

Let $A \subseteq S$ and *J* be an ideal of *S*. We define $A^{\perp_J} = \{x \in S : x \wedge a \in J \text{ for all } a \in A\}$. This is clearly a down set containing *J*. $A^{\perp_J}$ is called an annihilator of *A* relative to *J*. If it is an ideal, then it is called an annihilator ideal relative to *J*.

By [3,7] we know that, for any $a \in S$, $\{a\}^{\perp_J}$ is an ideal if and only if *S* is 0-distributive.

Following result is due to [6].

Lemma 2.1. *Let J be an ideal of a meet semilattice S. Suppose* $A, B \subseteq S$ *and* $a, b \in S$ *then the followinghold:*

(i) *If* $A \cap B = J$, *then* $B \subseteq A^{\perp_J}$.
(ii) $A \cap A^{\perp_J} = J$.
(iii) $A \subseteq B$ *implies that* $B^{\perp_J} \subseteq A^{\perp_J}$.
(iv) $a \leq b$ *implies that* $\{b\}^{\perp_J} \subseteq \{a\}^{\perp_J}$ *and* $\{a\}^{\perp_J \perp_J} \subseteq \{b\}^{\perp_J \perp_J}$
(v) $\{a\}^{\perp_J} \cap \{a\}^{\perp_J \perp_J} = J$
(vi) $\{a \wedge b\}^{\perp_J \perp_J} = \{a\}^{\perp_J \perp_J} \cap \{b\}^{\perp_J \perp_J}$
(vii) $A \subseteq A^{\perp_J \perp_J}$
(viii) $A^{\perp_J \perp_J \perp_J} = A^{\perp_J}$.

Homomorphism theorem for lattices can be found in Grätzer [4, Therem 11]. In a similar way, we can easily state the following homomorphism theorem for meet semilattices. We prefer to omit the proof as it is almost similar to the proof of homomorphism theorem for lattices.

Theorem 2.2. (*Homomorphism Theorem for meet semilattices*) *Every homomorphic image of a meet semilattice S is isomorphic to a suitable quotient meet semilattice of S. In fact, if* $\varphi : S \to T$ *is a homomorphism of S onto T and if* θ *is a congruence relation of S defined by* $x \equiv y(\theta)$ *if and only if* $\varphi(x) = \varphi(y)$. *Then* $S/\theta \cong T$.

Theorem 2.3. *Let S and T be two meet semilattice directed above.* I *is an ideal of S.* $f : S \to T$ *is a homomorphism and onto such that* $f^{-1}(f(I)) = I$. *Then* I *is semi prime in S implies* $f(I)$ *is semi prime in T.*

Proof: Suppose I is semi-prime. Let $x, y, z \in T$ with $x \wedge y \in f(I)$ and $x \wedge z \in f(I)$. Then there exists $a, b, c \in S$ such that $x = f(a),\ y = f(b),\ z = f(c)$. Now $f(a) \wedge f(b) = f(a \wedge b) \in f(I)$.
$f(a) \wedge f(c) = f(a \wedge c) \in f(I)$. This implies $a \wedge b, a \wedge c \in I$. Since I is semi prime, so there exists $d \in S$, $d \geq b, c$ such that $a \wedge d \in I$. Let $t = f(d)$.
Then $t = f(d) \geq f(b), f(c)$. That is $t \geq y, z$. Also $f(a) \wedge f(d) = f(a \wedge d) \in f(I)$.
Thus $x \wedge t \in f(I)$, and so $f(I)$ is semi prime.

Since S is 0-distributive if and only if $(0]$ is a semi prime ideal so the following corollary immediately follows by above theorem.

Corollary 2.4. *Let S and T be two directed above semilattices with 0.* $f : S \rightarrow T$ *is 0-homomorphism, onto and* $f^{-1}(0) = 0$. *Then T is 0-distributive if S is 0-distributive.*

Lemma 2.5. *Let* J *be a semi prime ideal of a directed above meet semilattice S.* $f : S \rightarrow \{\{a\}^{\perp_J \perp_j} : a \in S\}$ *given by* $f(a) = \{a\}^{\perp_J \perp_j}$. *Then the following results hold.*

i) f *is a meet homomorphism.*

ii) *For* $a \in S$, $f(a) = J$ *if and only if* $a \in J$.

iii) $f(\{a\}^{\perp_J}) = \{f(a)\}^{\perp_J}$.

Proof: (i) Let $a, b \in S$. Now

$$
\begin{aligned}
f(a \wedge b) &= \{a \wedge b\}^{\perp_J \perp_j} \\
&= \{a\}^{\perp_J \perp_J} \cap \{b\}^{\perp_J \perp_J} \\
&= f(a) \cap f(b) \\
&= f(a) \wedge f(b)
\end{aligned}
$$

Hence the map is a meet homomorphism.
(ii) If $f(a) = J$, then $\{a\}^{\perp_J \perp_j} = J$. Thus $\{a\}^{\perp_J} = \{a\}^{\perp_J \perp_J \perp_J} = S$ and so $a \in \{a\}^{\perp_J}$. This implies $a = a \wedge a \in J$.

Conversely, if $a \in J$, then $f(a) = \{a\}^{\perp_J \perp_J} = S^{\perp_J} = J$.

(iii) $f(\{a\}^{\perp_J}) = \{\{b\}^{\perp_J \perp_J} \mid b \in \{a\}^{\perp_J}\}$

$$
\begin{aligned}
f(\{a\}^{\perp_J}) &= \{\{b\}^{\perp_J \perp_j} \mid a \wedge b \in J\} \\
&= \{\{b\}^{\perp_J \perp_J} \mid f(a \wedge b) = J\} \quad \text{by(ii)} \\
&= \{\{b\}^{\perp_J \perp_J} \mid f(a) \wedge f(b) = J\} \\
&= \{f(a)\}^{\perp_J}
\end{aligned}
$$

Hence the proof is complete.

Corollary 2.6. *Let S be a 0-distributive meet semilattice S.* $f : S \to \{\{a\}^{\perp\perp} : a \in S\}$ *given by* $f(a) = \{a\}^{\perp\perp}$ *. Then the followings results hold.*

i) f *is a meet homomorphism.*

ii) *For* $a \in S$, $f(a) = \{0\}$ *if and only If* $a = 0$.

iii) $f(\{a\}^{\perp}) = \{f(a)\}^{\perp}$.

Note: Observe that lemma 2.5 is also true for an ordinary ideal *J* of *S*. But we have considered semi primeness of *J* as $\{a\}^{\perp_J}$ and $\{a\}^{\perp_J \perp_J}$ are ideals only when *J* is semi prime. Similarly, in a semilattice with 0, $\{a\}^{\perp}$ or $\{a\}^{\perp\perp}$ are ideals only when *S* is 0-distributive.

3. Weakly complemented semi lattices

Let *S* be a meet semilattic with 0. *S* is called weakly complemented if for any pair of distinct elements *a, b* of *S*, there exists an element c such that only one of $a \wedge c$ and $b \wedge c$ is equal to 0.

Similarly, for an ideal *J* of a meet semilattice *S*, we call *S* weakly complemented with respect to *J* if for any pair of distinct elements *a, b* of *S*, there exists an element *c* such that only one of $a \wedge c$ and $b \wedge c$ belongs to *J*. In particular, if $a < b$, then there exists $c \in S$ such that $a \wedge c \in J$ but $b \wedge c \notin J$.

Note that the definition of weakly complemented semilattice relative to ideal *J* can also be given in the following way:

For an ideal *J* of a meet semilattice *S, S* is called weakly complemented relative to *J* if for all $a, b \in S$, $a \neq b$ implies that either $\{a\}^{\perp_J} - \{b\}^{\perp_J} \neq \varphi$ or $\{b\}^{\perp_J} - \{a\}^{\perp_J} \neq \varphi$. These semilattices are also known as disjunctive semilattices relative to *J.*

Theorem 3.1. *Let S be a meet semilattice and J be a semi prime ideal of S. Then the following are equivalent;*

i) $f : S \to \{\{a\}^{\perp_J \perp_j} \mid a \in S\}$ *defined by* $f(a) = \{a\}^{\perp_J \perp_j}$ *is isomorphism.*

ii) $\{a\}^{\perp_J} = \{b\}^{\perp_J} \in I_J(S)$ *implies that* $a = b$ *for all* $a, b \in S$.

iii) *S is weakly complemented relative to J.*

Proof: (i)$\Rightarrow$(ii). Let $\{a\}^{\perp_J} = \{b\}^{\perp_J}$ and $a \neq b$. Then as f is an isomorphism, we have, $f(a) \neq f(b)$ which implies that $\{a\}^{\perp_J \perp_J} \neq \{b\}^{\perp_J \perp_J}$. Then there exists $x \in \{a\}^{\perp_J \perp_j}$ such that $x \notin \{b\}^{\perp_J \perp_j}$ which implies that $x \wedge z \notin J$ for some $z \in \{b\}^{\perp_J}$. Since $\{a\}^{\perp_J} = \{b\}^{\perp_J}$, then we have $x \wedge z \notin J$ for some $z \in \{a\}^{\perp_J}$ which implies $x \notin \{a\}^{\perp_J \perp_j}$. This gives is a contradiction. Hence $\{a\}^{\perp_J} = \{b\}^{\perp_J}$ implies $a = b$.

(ii) $\Rightarrow$(iii). Let $a < b$. Then by Lemma 1 and (ii), we have $\{a\}^{\perp_J} \supset \{b\}^{\perp_J}$. Hence there exists $x \in \{a\}^{\perp_J}$ such that $x \notin \{b\}^{\perp_J}$, which implies that *S* is weakly complemented relative to *J*.

(iii)$\Rightarrow$(ii). Let $a \neq b$ then either $a \wedge b < a$ or $a \wedge b < b$. Assume that $a \wedge b < a$. As *S* is weakly complemented, so there exists $x \in \{a \wedge b\}^{\perp_J}$ such that $x \wedge a \notin J$.

Thus we have $x \wedge (a \wedge b) \in J$. This implies $(x \wedge a) \wedge b \in J$, and so $(x \wedge a) \in \{b\}^{\perp_J}$ and $x \wedge a \notin \{a\}^{\perp_J}$.

Hence $\{a\}^{\perp_J} \neq \{b\}^{\perp_J}$, and so (ii) holds.

(ii) $\Rightarrow$ (i) . To prove f is an isomorphism. For all $a, b \in S$, $a = b$

$$\Leftrightarrow \{a\}^{\perp_J} = \{b\}^{\perp_J}$$
$$\Leftrightarrow \{a\}^{\perp_J \perp_J} = \{b\}^{\perp_J \perp_J}$$
$$\Leftrightarrow f(a) = f(b)$$

This implies f is well defined and one to one.

Obviously, the mapping is onto.

Moreover, by Lemma 2.5, f is a $\wedge$ preserving map. Therefore, *f* is an isomorphisom.

We conclude the paper with the following result as $(0]$is semi prime if and only if S is 0-distributive.

Corollary 3.2. *Let S be a 0-distributive meet semilattice. Then the following are equivalent;*

i) $f : S \to \{\{a\}^{\perp\perp} \mid a \in S\}$ *defined by* $f(a) = \{a\}^{\perp\perp}$ *is an isomorphism.*

ii) $\{a\}^{\perp} = \{b\}^{\perp} \in I(S)$ *implies that* $a = b$ *for all* $a, b \in S$.

iii) *S is weakly complemented.*

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