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A New Method for Computing the Inverses of Anti-Tridiagonal and Anti-Pentadiagonal Matrices

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Abstract. In this paper, we present an efficient method for solving the inverses of anti-tridiagonal and anti-pentadiagonal matrices draw support from symmetric circulant matrices. In addition, we establish the connections between anti-tridiagonal, anti-pentadiagonal matrices and symmetric circulant matrices Also some numerical examples are given.

Keywords: Anti-tridiagonal matrices; anti-pentadiagonal matrices; symmetric circulant matrices

AMS Mathematics Subject Classification (2010): 15A09, 15B05, 65F05

1. Introduction

It is well known that (anti-) tridiagonal and (anti-) pentadiagonal matrices are widely applied in applied mathematics and engineering mathematics. They are an effective tool in approximation theory, especially in the research of special functions and orthogonal polynomials [2,7]. Therefore, they also arise naturally in partial differential equations and numerical analysis [4,5,14,15]. In many of these areas, inverses of (anti-) tridiagonal and (anti-) pentadiagonal matrices are important, So a fast and efficient computational method to obtain the inverses of them is demanded. Of course, a large number of important methods have been posed, efficient algorithms [1,8] and explicit formula [3,6,12,15] for (k-) tridiagonal matrix inverse were presented. In [9] and [10], the authors presented recursive algorithm for inverting tridiagonal, ant-tridiagonal and pentadiagonal, anti-pentadiagonal matrices. In these methods, usually LU factorization is a main tool. What is new in our paper is to use symmetric circulant matrices for computing the inverse of them.

In this paper, we first consider the inverses of nonsingular ant-tridiagonal matrices with the following form

$$\begin{pmatrix} & & & a_1 & a_0 \\ & & \iddots & a_0 & a_{-1} \\ & \iddots & \iddots & \iddots & \\ a_1 & \iddots & \iddots & & \\ a_0 & a_{-1} & & & \end{pmatrix} \tag{1.1}$$

and the ant-pentadiagonal matrices as follows, respectively.

$$\begin{pmatrix} & & & & b_2 & b_1 & b_0 \\ & & & \iddots & b_1 & b_0 & b_{-1} \\ & & \iddots & \iddots & \iddots & \iddots & b_{-2} \\ b_2 & \iddots & \iddots & \iddots & \iddots & & \\ b_1 & \iddots & \iddots & \iddots & & & \\ b_0 & b_{-1} & b_{-2} & & & & \end{pmatrix} \tag{1.2}$$

We expand ant-tridiagonal and ant-pentadiagonal matrices to symmetric circulant matrices, and establish the connections between them and symmetric circulant matrices respectively. Also the same considerations occur to tridiagonal and pentadiagonal matrices with constant diagonals. Finally, we give some numerical examples and make some concluding conclusions.

2. Inverses of ant-tridiagonal and ant-pentadiagonal matrices

In this section, we give an approach to compute the inverses of ant-tridiagonal and ant-pentadiagonal matrices. We always assume that matrices discussed are nonsingular, unless otherwise stated. Suppose that A and B are ant-tridiagonal and ant-pentadiagonal matrices defined in (1.1), (1.2), respectively. We will assume $a_{-1}a_1 \neq 0$ and $b_{-i} \neq 0, b_i \neq 0, i = 1, 2$ to avoid trivial conditions. Let C be a symmetric circulant matrix with respect to $c_0, c_1, \cdots, c_{n-1}$, that is

$$C = \begin{pmatrix} c_0 & c_1 & c_2 & \cdots & c_{n-1} \\ c_1 & c_2 & c_3 & \cdots & c_0 \\ c_2 & c_3 & c_4 & \cdots & c_1 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ c_{n-1} & c_0 & c_1 & \cdots & c_{n-2} \end{pmatrix}, \tag{2.1}$$

we denote $C = \mathrm{scirc}(c_0, c_1, \cdots, c_{n-1})$.

First we give the following lemma.

Lemma 2.1. Let C be a symmetric circulant matrix of order n defined in (2.1). Then C^{-1} is also a symmetric circulant matrix. Suppose that $C^{-1} = scirc(d_0, d_1, \cdots, d_{n-1})$, then

$$d_i = \frac{1}{n}\sum_{k=0}^{n-1} \frac{1}{\sum\limits_{s=0}^{n-1} c_s\omega^{k(i-s)}}, i = 0, 1, \cdots, n-1. \tag{2.2}$$

Proof: For the sake of simplicity, we denote $C^* = \mathrm{scirc}(d_0, d_1, \cdots, d_{n-1})$, and c_{ij} stands for the element of i th row, j th column of CC^*. Thus we need only to prove that

$$c_{ij} = \begin{cases} 1, & i = j, \\ 0, & i \neq j. \end{cases}$$

It is easy to verify that

$$c_{ij} = \sum_{\substack{u,v=0 \\ u-v=i-j}}^{n-1} c_u d_v = \sum_{v=0}^{n-1} c_{v+n+i-j (\bmod n)} d_v,$$

where the symbol $p \,(\bmod q)$ denote the remainder of p divided by q. Thus we have

$$
\begin{aligned}
c_{ij} &= \sum_{v=0}^{n-1} c_{v+n+i-j(\bmod n)} \cdot \left(\frac{1}{n} \sum_{k=0}^{n-1} \frac{1}{\sum\limits_{s=0}^{n-1} c_s \omega^{k(v-s)}} \right) \\
&= \frac{1}{n} \sum_{k=0}^{n-1} \sum_{v=0}^{n-1} \frac{c_{v+n+i-j(\bmod n)} \cdot \omega^{-kv}}{\sum\limits_{s=0}^{n-1} c_s \omega^{-ks}} \\
&= \frac{1}{n} \sum_{k=0}^{n-1} \frac{\sum\limits_{v=0}^{n-1} c_{v+n+i-j(\bmod n)} \cdot \omega^{-kv}}{\sum\limits_{s=0}^{n-1} c_s \omega^{-ks}}.
\end{aligned}
$$

According to the relevant result in Number Theory and $\omega^n = 1$, we have

$$
\begin{aligned}
\omega^{-kv} &= \omega^{-kv(\bmod n)} = \omega^{-k(v+n+i-j)(\bmod n)} \cdot \omega^{k(i-j+n)(\bmod n)} \\
&= \omega^{-k(v+n+i-j)(\bmod n)} \cdot \omega^{k(i-j)(\bmod n)} \\
&= \omega^{-k(v+n+i-j)(\bmod n)} \cdot \omega^{k(i-j)}.
\end{aligned}
$$

Therefore

$$
\begin{aligned}
c_{ij} &= \frac{1}{n} \sum_{k=0}^{n-1} \frac{\sum\limits_{v=0}^{n-1} c_{v+n+i-j(\bmod n)} \cdot \omega^{-k(v+n+i-j(\bmod n))} \cdot \omega^{k(i-j)}}{\sum\limits_{s=0}^{n-1} c_s \omega^{-ks}} \\
&= \frac{1}{n} \sum_{k=0}^{n-1} \omega^{k(i-j)} \cdot \frac{\sum\limits_{v=0}^{n-1} c_{v+n+i-j(\bmod n)} \cdot \omega^{-k(v+n+i-j(\bmod n))}}{\sum\limits_{s=0}^{n-1} c_s \omega^{-ks}} \\
&= \frac{1}{n} \sum_{k=0}^{n-1} \omega^{k(i-j)} \cdot \frac{\sum\limits_{t=0}^{n-1} c_t \omega^{-kt}}{\sum\limits_{s=0}^{n-1} c_s \omega^{-ks}} = \frac{1}{n} \sum_{k=0}^{n-1} \omega^{k(i-j)}.
\end{aligned}
$$

If $i = j$, then

$$
c_{ij} = \frac{1}{n} \sum_{k=0}^{n-1} \omega^{k(i-j)} = \frac{1}{n} \cdot \sum_{k=0}^{n-1} 1 = 1.
$$

If $i \neq j$, then

$$
c_{ij} = \frac{1}{n} \sum_{k=0}^{n-1} \omega^{k(i-j)} = \frac{1}{n} \cdot \frac{1 - \omega^{n(i-j)}}{1 - \omega^{(i-j)}} = 0.
$$

This fact show that scric$(c_0, c_1, \cdots, c_{n-1}) \cdot$ scirc$(d_0, d_1, \cdots, d_{n-1}) = I_n$, as required. $\square$

For an ant-tridiagonal matrix A defined in (1.1), let

$$A_{sc} = \left(\begin{array}{c|ccc} a_{-1} & & & \\ \vdots & & A & \\ a_1 & & & \\ \hline a_0 & a_{-1} & \cdots & a_1 \end{array}\right) \tag{2.3}$$

Clearly, A_{sc} is a symmetric circulant matrix. According to Lemma 2.1 we have the following corollary.

Corollary 2.1. Let A_{sc} be a symmetric circulant matrix of order n and $A_{sc}^{-1} = scirc(s_0, s_1, \cdots, s_{n-1})$. Then

$$s_i = \frac{1}{n}\sum_{k=0}^{n-1} \frac{1}{\omega^{ki}(a_{-1} + a_0\omega^k + a_1\omega^{2k})}, i = 0, 1, \cdots, n-1. \tag{2.4}$$

Theorem 2.1. Suppose that A_{sc} is a $n \times n$ matrix defined in (2.3), A_{sc}^{-1} is partitioned as

$$A_{sc}^{-1} = \begin{pmatrix} \alpha^{\mathrm{T}} & a \\ G & \beta \end{pmatrix},$$

where $a \in \mathbb{C}$, $\alpha,\ \beta \in \mathbb{C}^{(n-1)\times 1}$, and $G \in \mathbb{C}^{(n-1)\times(n-1)}$. Then

$$A^{-1} = G - \frac{1}{a}\cdot \beta\alpha^{\mathrm{T}}. \tag{2.5}$$

Proof: For simplicity, we use 0 to denote the zero matrix whose sizes will be clear from the context. Let I_n denote the identity matrix of order n. By the identity $A_{sc}^{-1}\cdot A_{sc} = I_n$, we have

$$\begin{pmatrix} \alpha^{\mathrm{T}} & a \\ G & \beta \end{pmatrix}\begin{pmatrix} \gamma & A \\ a_0 & \gamma^{\mathrm{T}} \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & I_{n-1} \end{pmatrix},$$

where $\gamma = (a_{-1},\ 0, \cdots,\ a_1)^{\mathrm{T}} \in \mathbb{C}^{(n-1)\times 1}$. Then one can obtain that

$$\begin{aligned} \begin{pmatrix} \alpha^{\mathrm{T}} & a \\ G & \beta \end{pmatrix}\begin{pmatrix} \gamma & A \\ a_0 & \gamma^{\mathrm{T}} \end{pmatrix} &= \begin{pmatrix} \alpha^{\mathrm{T}}\gamma + aa_0 & \gamma^{\mathrm{T}}A + a\gamma^{\mathrm{T}} \\ G\gamma + a_0\beta & GA + \beta\gamma^{\mathrm{T}} \end{pmatrix} \\ &= \begin{pmatrix} 1 & 0 \\ 0 & I_{n-1} \end{pmatrix}. \end{aligned}$$

Comparing the entries of the both sides, we have

$$\begin{cases} \alpha^{\mathrm{T}}\gamma + aa_0 = 1, & (1) \\ \alpha^{\mathrm{T}}A + a\gamma^{\mathrm{T}} = 0, & (2) \\ GA + \beta\gamma^{\mathrm{T}} = I_{n-1}. & (3) \end{cases}$$

In the light of (1) and (2), we immediately attain that

$$\begin{pmatrix} \alpha^{\mathrm{T}} & a \\ I_{n-1} & 0 \end{pmatrix}\begin{pmatrix} \gamma & A \\ a_0 & \gamma^{\mathrm{T}} \end{pmatrix} = \begin{pmatrix} \alpha^{\mathrm{T}}\gamma + aa_0 & \alpha^{\mathrm{T}}A + a\gamma^{\mathrm{T}} \\ \gamma & A \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ \gamma & A \end{pmatrix}.$$

Taking the determinant of both sides, then $(-1)^{n+1}a|A_{sc}| = |A|$, which yields $a \neq 0$. By (2) and (3), we have

$$-\frac{1}{a}\beta\alpha^{\mathrm{T}}A + GA = I_{n-1},$$

as required. $\square$

According to Theorem 2.1, we give a relation between A^{-1} and the submatrices of A_{sc}^{-1}. Concretely speaking, in order to compute the inverse of A, we need only to obtain $a, \alpha^{\mathrm{T}}, \beta$ and G which are submatrices of A_{sc}^{-1}. Clearly

$$\alpha^{\mathrm{T}} = A_{sc}^{-1}(1,\, 1:n-1),\ \ \beta = A_{sc}^{-1}(2:n,\, n),\ \ G = A_{sc}^{-1}(2:n,\, 1:n-1),$$

where symbol $A_{sc}^{-1}(x_1 : x_2, y_1 : y_2)$ stands for the submatrix of A_{sc}^{-1} that lies on the intersection of rows $x_1, x_1+1, \cdots, x_2$ with columns $y_1, y_1+1, \cdots, y_2$.

Combining with Corollary 2.1 and Theorem 2.1, we immediately have

$$a = s_{n-1}, \alpha = \beta = \begin{pmatrix} s_0 \\ s_1 \\ \vdots \\ s_{n-2} \end{pmatrix}, G = \begin{pmatrix} s_1 & s_2 & s_3 & \cdots & s_{n-1} \\ s_2 & s_3 & s_4 & \cdots & s_0 \\ s_3 & s_4 & s_5 & \cdots & s_1 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ s_{n-1} & s_0 & s_1 & \cdots & s_{n-3} \end{pmatrix}. \tag{2.6}$$

Thus A^{-1} is given by (2.4), (2.5) and (2.6).

For an anti-pentadiagonal matrix B with the form (1.2), we let

$$B_{sc} = \left(\begin{array}{cc|ccccc} b_{-1} & b_{-2} & & & & & \\ b_{-2} & \vdots & & & & & \\ \vdots & \vdots & & & B & & \\ \vdots & b_2 & & & & & \\ b_2 & b_1 & & & & & \\ \hline b_1 & b_0 & b_{-1} & b_{-2} & \cdots & \cdots & b_2 \\ b_0 & b_{-1} & b_{-2} & \cdots & \cdots & b_2 & b_1 \end{array}\right). \tag{2.7}$$

According to Lemma 2.1 and (2.7), we have the following corollary.

Corollary 2.2. Let B_{sc} be a symmetric circulant matrix of order n defined in (2.7) and $B_{sc}^{-1} = scirc(t_0, t_1, \cdots, t_{n-1})$. Then

$$t_i = \frac{1}{n}\sum_{k=0}^{n-1} \frac{1}{\omega^{k(i-1)}(b_{-2} + b_{-1}\omega^k + b_0\omega^{2k} + b_1\omega^{3k} + b_2\omega^{4k})}, \tag{2.8}$$

for $i = 0, 1, \cdots, n-1$.

Theorem 2.2. Suppose that B_{sc} is a $n \times n$ matrix defined in (2.7), B_{sc}^{-1} is partitioned as

$$B_{sc}^{-1} = \begin{pmatrix} C & P \\ Q & D \end{pmatrix},$$

where $P \in \mathbb{C}^{2\times 2}$, $Q \in \mathbb{C}^{(n-2)\times(n-2)}$, $C \in \mathbb{C}^{2\times(n-2)}$, $D \in \mathbb{C}^{(n-2)\times 2}$. Then

$$B^{-1} = Q - DP^{-1}C. \tag{2.9}$$

Proof: Suppose that

$$B_{sc} = \begin{pmatrix} U & B \\ V & U^T \end{pmatrix}.$$

According to the equation $B_{sc}^{-1}B_{sc} = I_n$, we have

$$\begin{aligned} \begin{pmatrix} C & P \\ Q & D \end{pmatrix} \begin{pmatrix} U & B \\ V & U^T \end{pmatrix} &= \begin{pmatrix} CU + PV & CB + PU^T \\ QU + DV & QB + DU^T \end{pmatrix} \\ &= \begin{pmatrix} I_2 & 0 \\ 0 & I_{n-2} \end{pmatrix}. \end{aligned}$$

Therefore we obtain that

$$\begin{cases} CU + PV = I_2, \\ CB + PU^T = 0, \\ QB + DU^T = I_{n-2}. \end{cases}$$

Thus we have

$$\begin{pmatrix} C & P \\ I_{n-2} & 0 \end{pmatrix} \begin{pmatrix} U & B \\ V & U^T \end{pmatrix} = \begin{pmatrix} I_2 & 0 \\ U & B \end{pmatrix}.$$

Taking the determinant of both sides, then $|P| \cdot |B_{sc}| = |B|$, which shows that P is nonsingular. Since $CB + PU^T = 0$, then $U^T = -P^{-1}CB$. It follows that

$$QB - DP^{-1}CB = I_{n-2},$$

namely $B^{-1} = Q - DP^{-1}C$. This completes the proof. $\square$

From Theorem 2.2, we readily obtain the inverse of B by establishing a connection between B^{-1} and submatrices of B_{sc}. More specifically, in order to compute the inverse of B, we need only to obtain P, Q, C and D which are given by

$$\begin{aligned} P &= B_{sc}^{-1}(1:2, n-1:n), \quad & Q &= B_{sc}^{-1}(3:n, 1:n-2), \\ C &= B_{sc}^{-1}(1:2, 1:n-2), \quad & D &= B_{sc}^{-1}(3:n, n-1:n). \end{aligned}$$

Combining with Corollary 2.2 and Theorem 2.2, one can readily obtain

$$P = \begin{pmatrix} t_{n-2} & t_{n-1} \\ t_{n-1} & t_0 \end{pmatrix}, Q = \begin{pmatrix} t_2 & t_3 & t_4 & \cdots & t_{n-1} \\ t_3 & t_4 & t_5 & \cdots & t_0 \\ t_4 & t_5 & t_6 & \cdots & t_1 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ t_{n-1} & t_0 & t_1 & \cdots & t_{n-4} \end{pmatrix}, C^{\mathrm{T}} = D = \begin{pmatrix} t_0 & t_1 \\ t_1 & t_2 \\ \vdots & \vdots \\ t_{n-3} & t_{n-2} \end{pmatrix}. \quad (2.10)$$

Thus B^{-1} is determined by (2.8), (2.9) and (2.10).

3. Numerical examples and stability analysis

In this section, we give some numerical examples to support the theoretical analysis in Section 2.

Example 3.1. Now we consider the 5×5 anti-tridiagonal matrix A and 7×7 anti-pentadiagonal matrix $\widetilde{B}$, respectively.

$$A = \begin{pmatrix} & & & 4.2 & 0.5 \\ & & 4.2 & 0.5 & 2.7 \\ & 4.2 & 0.5 & 2.7 & \\ 4.2 & 0.5 & 2.7 & & \\ 0.5 & 2.7 & & & \end{pmatrix}, B = \begin{pmatrix} & & & & 2.2 & 4.5 & 1.2 \\ & & & 2.2 & 4.5 & 1.2 & 3.2 \\ & & 2.2 & 4.5 & 1.2 & 3.2 & -1.5 \\ & 2.2 & 4.5 & 1.2 & 3.2 & -1.5 & \\ 2.2 & 4.5 & 1.2 & 3.2 & -1.5 & & \\ 4.5 & 1.2 & 3.2 & -1.5 & & & \\ 1.2 & 3.2 & -1.5 & & & & \end{pmatrix}.$$

For the anti-tridiagonal matrix A, $n = 6$, $a_{-1} = 2.7$, $a_0 = 0.5$, $a_1 = 4.2$, $\omega = e^{-\pi \mathrm{i}/3}$. According to (2.4), (2.5) and (2.6), one can obtain

$$A^{-1} = \begin{pmatrix} 0.2838 & -0.0526 & -0.4317 & 0.1617 & 0.6417 \\ -0.0526 & 0.0097 & 0.0800 & -0.0299 & 0.2515 \\ -0.4317 & 0.0800 & 0.6568 & 0.1244 & -1.0447 \\ 0.1617 & -0.0299 & 0.1244 & 0.0236 & -0.1978 \\ 0.6417 & 0.2515 & -1.0447 & -0.1978 & 1.6617 \end{pmatrix}.$$

Similarly, for the anti-pentadiagonal matrix B, $n = 9$, $b_{-2} = -1.5$, $b_{-1} = 3.2$, $b_0 = 1.2$, $b_1 = 4.5$, $b_2 = 2.2$, $\omega = e^{-2\pi \mathrm{i}/9}$. The inverse of B is given by

$$B^{-1} = \begin{pmatrix} -0.1707 & 0.1226 & 0.1249 & -0.1474 & -0.0972 & 0.2292 & 0.1521 \\ 0.1226 & -0.0862 & -0.0859 & 0.1154 & 0.0986 & -0.0816 & 0.1251 \\ 0.1249 & -0.0859 & -0.0833 & 0.1283 & 0.1325 & 0.0094 & -0.2781 \\ -0.1474 & 0.1154 & 0.1283 & -0.0763 & 0.0699 & -0.0244 & -0.0368 \\ -0.0972 & 0.0986 & 0.1325 & 0.0699 & -0.2583 & 0.0469 & 0.2976 \\ 0.2292 & -0.0816 & 0.0094 & -0.0244 & 0.0469 & -0.0109 & -0.0451 \\ 0.1521 & 0.1251 & -0.2781 & -0.0368 & 0.2976 & -0.0451 & -0.3763 \end{pmatrix}.$$

Example 3.2. Consider the $n \times n$ anti-pentadiagonal matrix B_1 and B_2.

$$B_1 = \begin{pmatrix} & & & 1 & 0.6 & 2 \\ & & \iddots & 0.6 & 2 & 0.2 \\ & \iddots & \iddots & \iddots & \iddots & 0.1 \\ 1 & \iddots & \iddots & \iddots & \iddots & \\ 0.6 & \iddots & \iddots & \iddots & & \\ 2 & 0.2 & 0.1 & & & \end{pmatrix}_n, B_2 = \begin{pmatrix} & & & 0.3 & 0.8 & 4 \\ & & \iddots & 0.8 & 4 & 2 \\ & \iddots & \iddots & \iddots & \iddots & 1.2 \\ 0.3 & \iddots & \iddots & \iddots & \iddots & \\ 0.8 & \iddots & \iddots & \iddots & & \\ 4 & 2 & 1.2 & & & \end{pmatrix}_n.$$

What the follows we computer the errors $||I_n - B_iB_i^{-1}||_2$ when$n = 10, 35, 60$,95, 120,145, which are shown in Table 3.1 and Fig 3.1.

Table 3.1. The errors $||I_n - B_iB_i^{-1}||_2$

n	$\|\|I_n - B_1B_1^{-1}\|\|_2$	$\|\|I_n - B_2B_2^{-1}\|\|$
10	2.942941729886e-16	2.772457493531e-16
35	2.796285594189e-16	3.221554995895e-16
60	3.194018946232e-16	3.429558902270e-16
95	3.666606122158e-16	3.529665939961e-16
120	3.948950559207e-16	4.158004314647e-16
145	3.969365271431e-16	4.158004314678e-16

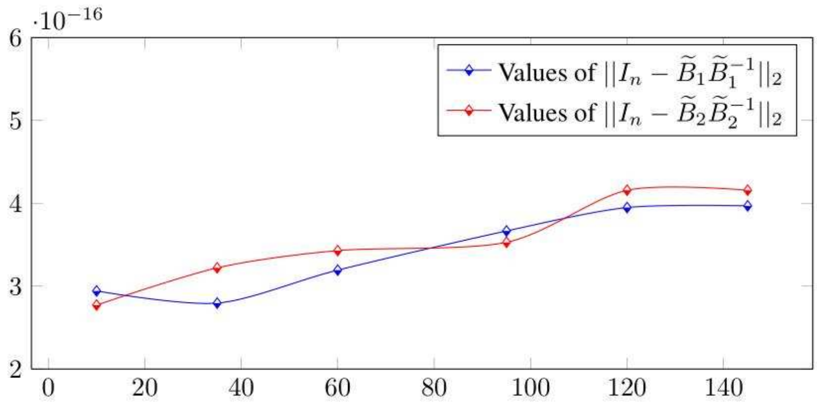

Figure 3.1: The errors $\left\|I_n - B_iB_i^{-1}\right\|_2$ changing with n

4. Conclusion

In this paper, we give an approach to compute anti-tridiagonal and anti-pentadiagonal matrices with constant anti-diagonals by means of the properties of symmetric circulant matrices. The advantages of our methods are steady and high-efficiency. More precisely, The complexity of the algorithm for anti-tridiagonal and anti-pentadiagonal matrices are $O(n\log_2^n)$ and $2n^2 + O(n\log_2^n)$, respectively. Furthermore, our methods can be applied to compute the inverse of anti-k-diagonal matrix with constant anti-diagonals, and a similar conclusion can be proved easily.

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