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Spectral Conditions for a Graph to be *k*-Connected

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Abstract. Using spectral radius and signless Laplacian spectral radius, we in this note present sufficient conditions for a graph to be k-connected.

Keywords: k-connected graph, spectral radius, signless Laplacian spectral radius

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1. Introduction

We consider only finite undirected graphs without loops or multiple edges. Notation and terminology not defined here follow those in [2]. For a graph $G = (V, E)$, we use n and e to denote its order $|V|$ and size $|E|$, respectively. We use $\delta = d_1 \leq d_2 \leq \cdots \leq d_n = \Delta$ to denote the degree sequence of a graph. The eigenvalues of a graph G are defined as the eigenvalues of its adjacency matrix $A(G)$. The largest eigenvalue, denoted $\rho(G)$, of a graph G is called the spectral radius of G. The signless Laplacian eigenvalues of a graph G are defined as the eigenvalues of the matrix $Q(G) := D(G) + A(G)$, where $D(G)$ is the diagonal matrix $diag(d_1, d_2, ..., d_n)$ and $A(G)$ is the adjacency matrix of G. The largest signless Laplacian eigenvalue, denoted $q(G)$, of a graph G is called the signless Laplacian spectral radius of G.

2. Main results

In [4], Li obtained sufficient conditions which are based on the spectral radius for some Hamiltonian properties of graphs. In [5], Li obtained sufficient conditions which are based on the signless Laplacian spectral radius for some Hamiltonian properties of graphs. Using similar ideas as the ones in [4] and [5], we will present sufficient conditions which are based on the spectral radius or the signless Laplacian spectral radius for a graph to be k-connected. The results are as follows.

Theorem 1. Let G be a connected graph of order $n \geq 2$ and let $1 \leq k \leq n-1$. If

$$\rho > \frac{n+k-5+\sqrt{(n+k-5)^2+8(n+k-3+(n-k-1)(\Delta-1))}}{4},$$

then G is k-connected.

Theorem 2. Let G be a connected graph of order $n \geq 2$ and let $1 \leq k \leq n-1$. If

$$q > \frac{(\Delta-1)+\sqrt{(\Delta-1)^2+8((n-2)\Delta+k-1)}}{2},$$

then G is k-connected.

Since $\Delta \le n-1$, Theorem 1 and Theorem 2 have the following Corollary 1 and Corollary 2, respectively.

Corollary 1. Let G be a connected graph of order $n \ge 2$ and let $1 \le k \le n-1$. If

$$\rho > \frac{n+k-5+\sqrt{(n+k-5)^2+8(n+k-3+(n-k-1)(n-2))}}{4},$$

then G is k- connected.

Corollary 2. Let G be a connected graph of order $n \ge 2$ and let $1 \le k \le n-1$. If

$$q > \frac{(n-2)+\sqrt{(n-2)^2+8((n-2)(n-1)+k-1)}}{2},$$

then G is k- connected.

In order to prove Theorem 1 and Theorem 2, we need the following results as our lemmas.

Lemma 1. ([1]) Let G be a graph of order $n \ge 2$ with degree sequence $d_1 \le d_2 \le \cdots \le d_n$ and let $1 \le k \le n-1$. If

$$1 \le i \le \lfloor \frac{n-k+1}{2} \rfloor, d_i \le i+k-2 \Rightarrow d_{n-k+1} \ge n-i,$$

then G is k - connected.

Lemma 2. ([6]) Let G be a connected graph with degree sequence $d_1 \le d_2 \le \cdots \le d_n$. Then for each i with $1 \le i \le n$,

$$\rho(G) \le \frac{d_i - 1 + \sqrt{(d_i+1)^2 + 4(i-1)(d_n - d_i)}}{2}.$$

Moreover, if $i = n$, the equality holds if and only if G is a regular graph. If $1 \le i \le n-1$, the equality holds if and only if G is either a regular graph or bidegreed graph in which $d_n = d_{n-1} = \cdots = d_{n-i+2} = n-1$ and $d_{n-i+1} = d_{n-i} = \cdots = d_1 = \delta$.

Lemma 3. ([7]) Let G be a connected graph with degree sequence $d_1 \le d_2 \le \cdots \le d_n$. Then for each i with $1 \le i \le n$,

$$q(G) \le \frac{d_n + 2d_i - 1 + \sqrt{(2d_i - d_n + 1)^2 + 8(i-1)(d_n - d_i)}}{2}.$$

Moreover, if $i = n$, the equality holds if and only if G is a regular graph. If $1 \le i \le n-1$, the equality holds if and only if G is either a regular graph or bidegreed graph in which $d_n = d_{n-1} = \cdots = d_{n-i+2} = n-1$ and $d_{n-i+1} = d_{n-i} = \cdots = d_1 = \delta$.

Lemma 4. ([3]) Let G be a graph of order n with maximum degree Δ. Then

$$q(G) \le 2\Delta.$$

Moreover, if G is connected, then equality holds if and only if G is regular.

Proof of Theorem 1. Let G be a graph satisfying the conditions in Theorem 1. Suppose that G is not k - connected. Then, from Lemma 1, there exists an integer j such that $1 \le j \le \lfloor \frac{n-k+1}{2} \rfloor \le \frac{n-k+1}{2}$, $d_j \le j+k-2$, and $d_{n-k+1} \le n-j-1$. Obviously, $d_j \ge 1$. Let $i = j$ in Lemma 2. Then we have that

$$\rho \le \frac{d_j - 1 + \sqrt{(d_j+1)^2 + 4(j-1)(d_n - d_j)}}{2}.$$

Thus

$$\rho^2 \le \rho(d_j - 1) + d_j + (j-1)(d_n - d_j).$$

Therefore

$$\rho^2 \le \rho(j+k-3) + j + k - 2 + (j-1)(\Delta - 1).$$

Hence

$$\rho^2 \le \rho\left(\frac{n-k+1}{2} + k - 3\right) + \frac{n-k+1}{2} + k - 2 + \left(\frac{n-k+1}{2} - 1\right)(\Delta - 1).$$

By solving the inequality, we have that

$$\rho \le \frac{n+k-5+\sqrt{(n+k-5)^2 + 8(n+k-3+(n-k-1)(\Delta-1))}}{4},$$

which is a contradiction.

This completes the proof of Theorem 1. □

Proof of Theorem 2. Let G be a graph satisfying the conditions in Theorem 2. Suppose that G is not k-connected. Then, from Lemma 1, there exists an integer j such that $1 \le j \le \lfloor \frac{n-k+1}{2} \rfloor \le \frac{n-k+1}{2}$, $d_j \le j+k-2$, and $d_{n-k+1} \le n-j-1$. Obviously, $d_j \ge 1$. Let $i = j$ in Lemma 3. Then we have that

$$q \le \frac{d_n + 2d_j - 1 + \sqrt{(2d_j - d_n + 1)^2 + 8(j-1)(d_n - d_j)}}{2}.$$

Thus

$$q^2 \le q(d_n + 2d_j - 1) + 2d_j(1 - d_n) + 2(j-1)(d_n - d_j).$$

Therefore,

$$q^2 \le q(d_n - 1) + 2d_j(q - d_n + 1) + 2(j-1)(d_n - d_j).$$

By Lemma 4, we have that

$$q^2 \le q(\Delta-1)+2(j+k-2)(\Delta+1)+(n-k-1)(\Delta-1).$$

Hence

$$q^2 \le q(\Delta-1)+(n+k-3)(\Delta+1)+(n-k-1)(\Delta-1).$$

By solving the inequality, we have that

$$q \le \frac{(\Delta-1)+\sqrt{(\Delta-1)^2+8((n-2)\Delta+k-1)}}{2},$$

which is a contradiction.

This completes the proof of Theorem 2. □

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