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Connectivity Analysis of a Network by Levels

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Abstract. In a weighted graph model, the reduction of flow value between some pairs of nodes is more relevant and more frequent than the total disruption of the flow or the disconnection of the entire network. So it is necessary to analyse connectivity by levels. Some new connectivity and acyclicity parameters are introduced in this paper and weighted trees are categorized depending on the structure of the level graphs.

Keywords: weighted graph, partial cutnode, strength reducing set

AMS Mathematics Subject Classifications (2010): 05C22, 05C38, 05C40

1. Introduction

We can consider a totally weighted graph (Both nodes and arcs are given weights) G as a pair (σ,μ) where $\sigma : V \to \Re$ and $\mu : V\times V \to \Re$. Also we denote the underlying graph by $G:(\sigma^*,\mu^*)$, where $\sigma^* = \{u \in V, \sigma(u) > 0\}$ and $\mu^* = \{(u,v) \in V\times V;\ \mu(u,v) > 0\}$ [11]. If we have a weighted graph representing some cities and roads connecting them, then μ may be taken as the populations function and σ the daily exchange of population between the cities. In such a model, connectivity plays a crucial role. Similar problems arise in all types of networks like communication, computer, biological, etc. Depending on the strength of a vertex (population, size, capacity), we can have different categories of vertices (A Class city, Big dam, etc). Also depending on the rate of flow (data flow, transport, etc.) between two nodes, we can have different categories of arcs (Broad band, national highway, etc.). Producing a path between any two nodes guarantee the connectedness of a network. But in any network, the analysis by different levels is very important. For example, high exchange of people between two big cities, less flow rate of data in a network of large capacity, etc. Also it is relevant to verify whether nodes of a given level are connected by roads of sufficient higher level. If higher category nodes are connected only by law category arcs or vice versa, the design is likely to be defective.

In this article we introduce some new connectivity concepts in weighted graphs. In a weighted graph model, for example, in an information network or electric circuit, the reduction of flow between pairs of nodes is more relevant and may frequently occur than the total disruption of the flow or the disconnection of the entire network [10,12]. This concept is our motivation. As weighted graphs are generalized structures of graphs, the concept introduced in this article also generalizes the classic connectivity concepts. Some related works can be seen in [7,8,9].

A *weighted graph G* is a graph in which every arc *e* is assigned a nonnegative number $w(e)$, called the weight of *e*. A graph is said to be totally weighted if both its node set and arc set are weighted [8]. The set of all the neighbours of a node *v* in *G* is denoted by $N_G(v)$ or simply $N(v)$, and its cardinality by $d_G(v)$ or $d(v)$[4]. The weighted degree of *v* is defined as $d_G^w(v) = \sum_{x \in N(v)} w(v, x)$. When no confusion occurs, we denote $d_G^w(v)$ by $d^w(v)$. The weight of a cycle is defined as the sum of the weights of its arcs. An unweighted graph can be regarded as a weighted graph in which every arc *e* is assigned weight $w(e) = 1$. Thus, in an unweighted graph, $d^w(v) = d(v)$ for every node *v*, and the weight of a cycle is simply the length of the cycle. An optimal cycle is a cycle which has maximum weight [1].

The *strength* of a path *P* of *n* edges e_i, for $1 \le i \le n$, denoted by $s(P)$ is equal to $s(P) = \min_{1 \le i \le n} \{w(e_i)\}$ [8]. The *strength of connectedness* of a pair of vertices $u, v \in V(G)$, denoted by $CONN_G(u, v)$, is defined as $CONN_G(u, v) = \{\text{Max } s(P); P \text{ is a } u\text{-}v \text{ path in } G\}$. If *u* and *v* are in different components of *G*, then $CONN_G(u, v) = 0$. A u-*v* path in a weighted graph G is called a *strongest u*-*v* path if $s(P) = CONN_G(u, v)$. An edge (x, y) is *strong* if its weight is at least equal to the strength of connectedness between *x* and *y* in *G*. [8] A connected weighted graph is called a *partial tree* if *G* has a spanning sub graph *F* $(V; E)$ which is a tree, where for all arcs (x,y) of *G* which are not in *F* , we have $CONN_G(x, y) > \mu(x, y)$.

Let G: (σ, μ) be a weighted graph, then for any two nodes *u* and *v* of *G*, the $\theta-$evaluation of *u* and *v* is defined as $\theta(\text{u, v}) = \{\alpha; \alpha \in \Re\}$, where α is the strength of a strong cycle passing through both *u* and *v* [14]. Max $\{\alpha; \alpha \in \theta(u, v); u, v \in \sigma^*\}$ is defined as the cycle connectivity between *u* and *v* in *G* and is denoted by $C_{u,v}{}^G$. Cycle connectivity of a graph *G* is defined as $CC(G) = \max\{C_{u,v}^G; u, v \in \sigma^*\}$.

2. t-cuts and connectedness levels

Consider a totally weighted graph $G:(\sigma, \mu)$, For $t \in \Re$, the direct *t*-cut of *G* or *t*-cut of *G*, denoted by G^t or (σ^t, μ^t) is the sub graph of *G* with node set $\sigma^t = \{x \in V / \sigma(x) \ge t\}$ and arc set $\mu^t = \{(x, y) \in V \times V / \mu(x, y) \ge t\}$. Consider Example 1.

Even though both the structures in figure 2 are the same, the connectivity of G_2 is more than that of G_1. The strength of a path in G_2 is 8 where as that of G_1 is 1. Now consider another example (Figure 3).

All *t*-cuts G^t of G_1 in (Figure 3) are connected for $t \in (4,8]$, but not in G_2. Thus G_1 shows some kind of connectedness which is not in G_2. Motivated by the above examples, we have the following definition.

Definition 1. *Let G be a weighted graph. Then the connectedness level of G is defined as*

$C(G) = \min\{CONN_G(x, y) / x, y \in V, x \neq y\}$.

Example 1. (Figure 1)

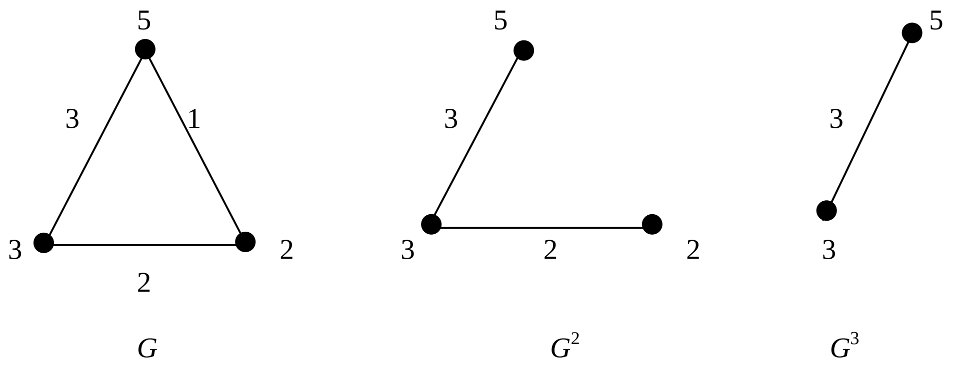

Figure 1: *t*-cuts of a graph

Now consider the following two graphs.

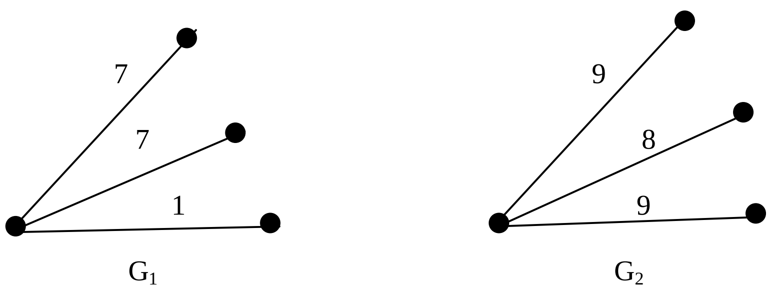

Figure 2: Different connectivity levels.

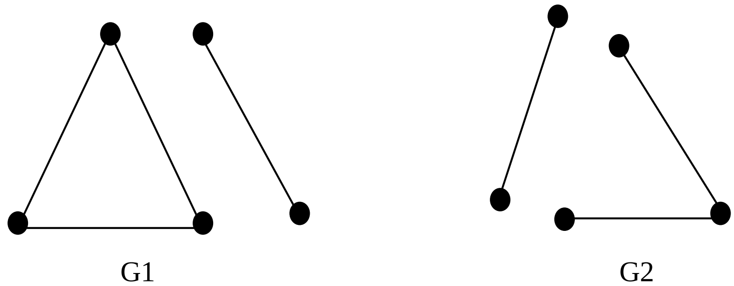

Figure 3: Connected *t*-cuts.

Note that if $C(G) = 0$, then only G is connected. Also if $C(G) = t$, then G^t will be connected.

Definition 2. *Let G be a weighted graph. Then G is called weakly connected if there exists some t-cut of G which is connected.*

In the previous example, G_1 is weakly connected. Clearly every connected weighted graph G is weakly connected. But converse is not true. (G_1 in previous example is weakly connected but not connected.)

Given a graph *G*, the *cyclomatic number* of *G* is defined as *m*- *n* + *p* where n, m and p denote the number of vertices, number of edges and number of connected components respectively. Using this we have a new definition as follows.

Definition 3. [9] *Let G be a weighted graph with n vertices, m edges and p connected components. The cyclomatic function* $\psi(G,t):\Re \to N \cup \{0\}, t \in \Re$ *is defined by* $\psi(G,t)$ *= cyclomatic number of* G^t = $m^t - n^t + p^t$, *where* n^t, m^t, p^t *denotes the number of vertices strong edges and connected components of* G^t.

Example 2: Consider the following weighted graph (Figure 4) with two components, 5 vertices and 5 edges.

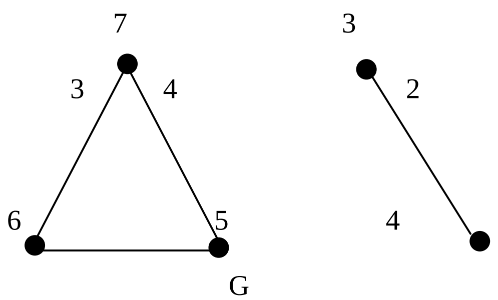

Figure 4: Cyclomatic number of a disconnected graph

Here for any $t \in (-\infty,2)$, $h(G,t) = m^t - n^t + p^t = 4 - 5 + 2 = 1.$ For *t* =2.1, $h(G,2.1) = 3 - 5 + 3 = 1, etc.$

Now we shall discuss some of the elementary properties of the cyclomatic function $\psi(G,t)$, First we have a trivial proposition.

Proposition 1. For any totally weighted graph *G*, and $t \in \Re$, $\psi(G,t) \geq 0..$

Also note that *h* is a piecewise constant function with finitely many jumps. In the next theorem, we show that ψ is a non increasing function.

Theorem 1. Let G be a totally weighted graph and let $t, t' \in \Re$ such that $t \geq t'$. Then $\psi(G,t) \leq \psi(G,t')$. That is ψ is non increasing in t.

Proof: We have $m^t \leq m^{t^I}$ implies $m^t = m^{t^I} - k_1$, $k_1 \in \mathrm{Z}$ and $k_1 \geq 0$. Also $n^t \leq n^{t^I}$ implies $m^t = m^{t^I} - k_2$, $k_2 \in \mathrm{Z}$ and $k_2 \geq 0$. The number of connected components of G^t may increase, decrease or remains same as t ranges over $\Re$. Thus for $t \geq t^{'}$, $p^t = p^{t^I} - k_3$, for some $k_3 \in \Re$ and $k_3 \geq 0$.

Thus $\psi(G,t) = (m^{t^I} - k_1)$-$(n^{t^I} - k_2)$ + $(p^{t^I} - k_3)$ = $\psi(G,t^{I}) + (k_2 - k_1 - k_3) =$ $\psi(G,t^{I}) + k$. We shall prove that k > 0 is impossible. If $k_2 < 0$ there is nothing to prove. So we prove the following cases.

Case 1. $k_2 = 0$.

This case implies that the vertex set does not change from G^t to G^{t^I}. Thus $k_3 \leq 0$, because components cannot decrease by edge suppression. Thus $-k_3 \leq k_1$, which implies $k \leq 0$.

Case 2. $k_2 > 0$.

We denote *h*, the number of eliminated connected components of G^{t^I}.. Obviously $k_3 \leq h$ and $k_2 \leq h$. Let $h = k_2 - s$; $s \in \{0,1,2,...,k_2\}$. From the definition of *s*, $k_1 \geq s$ and thus $k = k_2 - k_3 - k_1 \leq k_2 - k_3 + s - k$, which implies $k \leq s - k_1 \leq 0$.

Next we have a definition.

Definition 4. The cyclomatic kernel of a weighted graph *G* is defined as $K_a(G) = \{t \in \Re; \psi(G,t) = 0\}$.

In other words, $K_a(G)$ is the set of all real numbers such that G^t is a forest. In the previous example, $5 \in K_a(G)$. Using this kernel, we have a measure for the acyclic nature of the graph.

Definition 5. The strong acyclic level of a weighted graph *G* is defined as $S_w(G) = \inf\{t; t \in K_a(G)\}$ and which is same as the cycle connectivity of the weighted graph *G*. i.e. $S_w(G) = CC(G)$. It will help to measure the number of strong cycles in a given graph *G*.

The following theorem can be easily proved from the above definition and by using the properties of cycle connectivity.

Theorem 3. Let $G:(\sigma,\mu)$ be a weighted graph, then G^t has no cycles if and only if $t > CC(G)$.

Proof: Let $G:(\sigma,\mu)$ be a connected weighted graph such that G^t has no cycles in it. Let $CC(G)$ be the cycle connectivity of G such that $C_{u;v}{}^{G}$ < t for every $u,v \in V$. For otherwise suppose that there exist a cycle in G, whose strength is more than or equal to t. which will be remain in G^t. which is a contradiction. Conversely, suppose $CC(G)$ be the cycle connectivity of G such that $CC(G)$ < t, to prove G^t has no cycles. Assume that G^t has a cycle (say) C. Then strength of the cycle $C(s(C))$ is more than or equal to t, therefore $CC(G) \geq s(C) \geq t,$ again a contradiction. This completes the proof.

Now we can introduce a definition using cycle connectivity.

Definition 6. A weighted graph $G:(\sigma,\mu)$ is said to be fully strong acyclic if and only if CC(G)=0.

Clearly a weighted graph is fully acyclic if and only if it is a partial forest. Let G_1, G_2, …, G_n be the components of G, then for each G_i, i= 1,2,…,n is a partial tree. Then by the theorem 2.7 in [15], $CC(G)$=0.

Definition 7. A weighted graph $G:(\sigma,\mu)$ is strong acyclic by t-cuts if there exist a $t \in \Re$ such that G_t has no cycles.

Proposition 2. Every strongly acyclic graph is strong acyclic by t- cuts.
Proof: Let $G:(\sigma,\mu)$ be any strongly acyclic weighted graph, then $CC(G)$ = 0 and hence $C_{u,v}{}^{G}$ = 0 for every pair of nodes in G. That is G will be strong acyclic by t-cuts.

3. Concluding remarks

Connectivity concepts are the key in graph clustering and network problems. The classical parameters are dealing with the disconnection of the graph. In practical applications the reduction in the flow is more frequent than the disconnection. The authors made an attempt to generalize the connectivity concepts in weighted graphs. Also one of the major theorems in Graph theory due to Whitney is generalized.

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